11. Monday, Sept. 22

Last time, we were talking about homeomorphisms.

Example 11.1. (1) Consider $\tan : (0, \frac{\pi}{2}) \longrightarrow (0, \infty)$. This is a continuous bijection with continuous inverse (given by arctangent)

(2) Consider $\ln : (0, \infty) \longrightarrow \mathbb{R}$. This is a continuous bijection with inverse e^x. Composing homeomorphisms produces homeomorphisms, and we therefore get a homeomorphism

$$(0,1) \xrightarrow{\cong} (0, \frac{\pi}{2}) \xrightarrow{\cong} (0, \infty) \xrightarrow{\cong} \mathbb{R}.$$

(3) We similarly get a homeomorphism $\tan : [0, \frac{\pi}{2}) \xrightarrow{\cong} [0, \infty)$. It follows that we have

$$[0,1) \cong [0, \infty) \qquad \text{and} \qquad (0,1] \cong [0, \infty).$$

(4) One can similarly get $B_r^n(x) \cong \mathbb{R}^n$ for any n, r, and x.

The above example shows that there really are only three intervals, up to homeomorphism: the open interval, the half-open interval, and the closed interval.

We say that two spaces are **homeomorphic** if there is a homeomorphism between them (and write $X \cong Y$ as above). This is the notion of "sameness" for spaces. One of the major overarching questions for this course will be: how can we tell when two spaces are the same or are actually different?

A standard way to show that two spaces are not homeomorphic is to find a property that one has and the other does not. For instance every metric space is Hausdorff, so no non-Hausdorff space is the "same" as a metric space. But what property distinguishes the 3 interval types above? As we learn about more and more properties of spaces, this question will become easier to answer.

In the exponential example from last time, we noted that homeomorphisms must take open sets to open sets. Such a map is called an **open map**. Similarly, a **closed map** takes closed sets to closed sets.

Proposition 11.2. *Let $f : X \longrightarrow Y$ be a continuous bijection. The following are equivalent:*

(1) *f is a homeomorphism*
(2) *f is an open map*
(3) *f is a closed map*

If we drop the assumption that f is bijective, it is no longer true that being an open map is equivalent to being a closed map. For example, the inclusion $(0,1) \longrightarrow \mathbb{R}$ is open but not closed, and the inclusion $[0,1] \longrightarrow \mathbb{R}$ is closed but not open.

Put on your hard hats! We turn now to the construction phase. We considered the product of metric spaces: let's define the product for spaces. We already know what property it should satisfy: we want it to be true that mapping continuously from some space Z into the product $X \times Y$ should be the same as mapping separately to X and to Y. Another way to describe this is that we want $X \times Y$ to be the "universal" example of a space with a pairs of maps to X and Y.

Well, if the projection $p_X : X \times Y \longrightarrow X$ is to be continuous, we need $p_X^{-1}(U) = U \times Y$ to be open whenever $U \subseteq X$ is open. Similarly, we need $X \times V$ to be open if $V \subseteq Y$ is open. We are forced to include these open sets, but we don't want to throw in anything extra that we don't need. In other words, we want the product topology on $X \times Y$ to be the coarsest topology containing the sets $U \times Y$ and $X \times V$.

Note that if we consider the collecion

$$\mathcal{B} = \{U \times Y\} \cup \{X \times V\},$$

this cannot be a basis because it fails the intersection property. A typical intersection is

$$(U \times Y) \cap (X \times V) = U \times V,$$

and if we consider all sets of this form, we do get a basis.

Definition 11.3. Given spaces X and Y, the **product topology** on $X \times Y$ has basis given by sets of the form $U \times V$, where U and V are open in X and Y, respectively.

This satisfies the universal property of a product. We have engineered the definition to make this so, but we will check this anyway. First, we make a little detour.

We pointed out above that if we considered the collection

$$\mathcal{B} = \{U \times Y\} \cup \{X \times V\},$$

we would not have a basis, as the intersection property failed. We remedied this by considering instead intersections of elements of $\mathcal{B}$. This is a useful idea that shows up often.

Given a set X, a collection $\mathcal{C}$ of subsets of X is called a **prebasis** for a topology on X if the collection covers X. Actually, in all of the textbooks, this is called a subbasis, but that is a terrible name, since it suggests that it is a basis. I will try to stick with the better name of prebasis.

We can then get a basis from the prebasis by considering finite intersections of prebasis elements.

Example 11.4. The collection of rays (a, ∞) and $(-\infty, b)$ give a prebasis for the standard topology on $\mathbb{R}$.

We introduced the product topology above and mentioned the universal property, but let's spend a little bit of time with it to really nail down the concept.

Theorem-Definition 11.5. *Let X and Y be spaces. Then $X \times Y$, together with the projection maps*

$$p_X : X \times Y \longrightarrow X \quad \text{and} \quad p_Y : X \times Y \longrightarrow Y,$$

satisfies the following "universal property": given any space Z and maps $g : Z \longrightarrow X$ and $h : Z \longrightarrow Y$, there is a unique continuous map $f : Z \longrightarrow X \times Y$ such that

$$g = p_X \circ f, \qquad h = p_Y \circ f.$$

$$\begin{tikzcd}
& & X \\
Z \arrow[r, dashed, "\exists! f"] \arrow[urr, bend left, "g"] \arrow[drr, bend right, "h"'] & X \times Y \arrow[ur, "p_X"'] \arrow[dr, "p_Y"] & \\
& & Y
\end{tikzcd}$$

Proof. The uniqueness is clear: if there exists such a continuous map f, then the conditions force this to be $f = (g, h)$. The only question is whether or not f is continuous. Consider a typical basis element $U \times V$ for the product topology on $X \times Y$. Then

$$\begin{aligned} f^{-1}(U \times V) = \{z \in Z \mid f(z) \in U \times V\} &= \{z \in Z \mid g(z) \in U \text{ and } h(z) \in V\} \\ &= g^{-1}(U) \cap h^{-1}(V), \end{aligned}$$

which is an intersection of open sets and therefore open. ∎

Ok, so we showed that $X \times Y$ satisfies this property, but why do we call this a "universal property"?

Proposition 11.6. *Suppose W is a space with continuous maps $q_X : W \longrightarrow X$ and $q_Y : W \longrightarrow Y$ also satisfying the property of the product. Then W is homeomorphic to $X \times Y$.*

Proof. The universal property for $X \times Y$ gives us a map $f : W \longrightarrow X \times Y$.

$$\begin{tikzcd}
 & & X \\
W \arrow[r, dashed, "\exists! f"] \arrow[urr, bend left, "q_X"] \arrow[drr, bend right, "q_Y"'] & X \times Y \arrow[ur, "p_X"'] \arrow[dr, "p_Y"] & \\
 & & Y
\end{tikzcd}$$

But W also has a universal property, so we get a map $\varphi : X \times Y \longrightarrow W$ as well.

$$\begin{tikzcd}
 & & X \\
X \times Y \arrow[r, dashed, "\exists! \varphi"] \arrow[urr, bend left, "p_X"] \arrow[drr, bend right, "p_Y"'] & W \arrow[ur, "q_X"'] \arrow[dr, "q_Y"] & \\
 & & Y
\end{tikzcd}$$

Now make Pacman eat Pacman!

$$\begin{tikzcd}
 & & & X \\
W \arrow[r, dotted, "f"] \arrow[urrr, bend left, "q_X"] \arrow[drrr, bend right, "q_Y"'] & X \times Y \arrow[r, dotted, "\varphi"] \arrow[urr, dotted, "p_X"] \arrow[drr, dotted, "p_Y"'] & W \arrow[ur, "q_X"'] \arrow[dr, "q_Y"] & \\
 & & & Y
\end{tikzcd}$$

We have a big diagram, but if we ignore all dotted lines, there is an obvious horizontal map $W \longrightarrow W$ to fill in the diagram, namely the id_W. Since the universal property guarantees that there is a **unique** way to fill it in, we find that $\varphi \circ f = \mathrm{id}_W$. Reversing the pacmen gives the other equality $f \circ \varphi = \mathrm{id}_{X \times Y}$. In other words, f is a homeomorphism, and $\varphi = f^{-1}$. ∎

This argument may seem strange the first time you see it, but it is a typical argument that applies any time you define an object via a universal property. The argument shows that any two objects satisfying the universal property must be "the same".

Ok, so we understand $X \times Y$ as a topological space. What about a product of more than two spaces? Well, if we have a finite collection $X_1, \ldots, X_n$ of spaces, the product topology on $X_1 \times \cdots \times X_n$ has basis given by the $U_1 \times \cdots \times U_n$, or equivalently, prebasis given by the $p_j^{-1}(U_j)$. Note that this is equivalent because the basis element $U_1 \times \cdots \times U_n$, is a finite intersection of the prebasis elements $p_j^{-1}(U_j)$.

But what about the product of an *arbitrary* number of spaces? Here, the property we want is that whenever we have a space Z and maps $f_j : Z \longrightarrow X_j$ for all i, then there should be a unique continuous map $f : Z \longrightarrow \prod_{j \in J} X_j$ such that $p_j \circ f = f_j$.

Just as for finite products, we want the projection maps $p_j : \prod_{j \in J} \longrightarrow X_j$ to be continuous. This forces each $p_j^{-1}(U_j)$ to be continuous, and we can again choose these for a prebasis. We thus get a basis consisting of finite intersections $p_{j_1}^{-1}(U_{j1}) \cap \cdots \cap p_{j_k}^{-1}(U_{j_k})$.

Definition 11.7. Given spaces X_j, one for each $j \in J$, the **product topology** on $\prod_{j\in J} X_j$ has basis consisting of the $p_{j_1}^{-1}(U_{j1}) \cap \cdots \cap p_{j_k}^{-1}(U_{j_k})$.

12. Wed, Sept. 27

On the homework that was just returned, a few people used the fact that a product of continuous maps is continuous. This is true, but we have not discussed it yet, so let's do that now.

Proposition 12.1. *Let $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ be continuous. Then the product map $f \times f' : X \times X' \longrightarrow Y \times Y'$ is also continuous.*

Proof. This follows very easily from the universal property. If we want to map continuously to $Y \times Y'$, it suffices to specify continuous maps to Y and Y'. The continuous map $X \times X' \longrightarrow Y$ is the composition

$$X \times X' \xrightarrow{p_X} X \xrightarrow{f} Y,$$

and the other needed map is the composition

$$X \times X' \xrightarrow{p_{X'}} X' \xrightarrow{f'} Y'.$$

■

Last time, we introduced the *product topology* on $\prod_{\alpha\in A} X_j$, which had basis

$$\mathcal{B}_{\text{prod}} = \left\{ \prod_{\alpha} U_\alpha \mid U_\alpha \subseteq X_\alpha \text{ is open, and only finitely many } U_\alpha \text{ are } \textbf{proper} \text{ subsets} \right\}.$$

Proposition 12.2. *The product topology on $\prod_{\alpha\in A} X_\alpha$, as defined above, satisfies the following universal property: given any space Z and continuous maps $f_\alpha : Z \longrightarrow X_\alpha$ for all $\alpha \in A$, there is a unique continuous $f : Z \longrightarrow \prod_{\alpha\in A} X_\alpha$ such that $p_\alpha \circ f = f_\alpha$ for all $\alpha \in A$.*

Proof. The same proof as that given in 11.5 works here. Given the maps f_α, we define f by $f(z)_\alpha = f_\alpha(z)$. Again, the equations $p_\alpha \circ f = f_\alpha$ force this choice on us. The only question is whether this makes f into a continuous map. Since the topology on $\prod_{\alpha\in A} X_\alpha$ is defined by the prebasis elements $p_\alpha^{-1}(U_\alpha)$, it suffices to show that each of these pulls back to an open set. But

$$f^{-1}(p_\alpha^{-1}(U_\alpha)) = (p_\alpha \circ f)^{-1}(U_\alpha) = f_\alpha^{-1}(U_\alpha),$$

which is open since f_α is continuous. ■

But there is another obvious guess, coming from the answer for finite products. We can think about the basis consisting of products $\prod_{\alpha} U_\alpha$. This is no longer equivalent to the product topology!

Definition 12.3. Suppose given a collection of spaces X_α. The **box topology** on $\prod_{\alpha\in A} X_\alpha$ is generated by the basis $\left\{ \prod_{\alpha\in A} U_\alpha \right\}$.

As discussed above, the box topology has more open sets; in other words, the box topology is finer than the product topology. To see that the box topology does not have the universal property we want, consider the following example: let $\Delta : \mathbb{R} \longrightarrow \prod_{n\in\mathbb{N}} \mathbb{R}$ be the diagonal map, all of whose component maps are simply the identity. For each n, let $I_n = (\frac{-1}{n}, \frac{1}{n})$. In the box topology, the subset $I = \prod_n I_n \subseteq \prod_n \mathbb{R}$ is an open set, but

$$\Delta^{-1}(I) = \bigcap_n \mathrm{id}^{-1}(I_n) = \bigcap_n I_n = \{0\}$$

is not open. So the diagonal map is not continuous in the box topology!

Since we are now considering arbitrary products, it may be useful to stop and clarify what we mean. For instance, we might want to consider a countable infinite product $\mathbb{R} \times \dots$.

Let X_α, for $\alpha \in A$, be sets. The cartesian product $\prod_{\alpha\in A} X_\alpha$ is the collection of tuples (x_α), where $x_\alpha \in X_\alpha$. This means that for each $\alpha \in A$, we want an element $x_\alpha \in X_\alpha$. In other words, we should have a function

$$x_- : A \longrightarrow X = \bigcup_\alpha X_\alpha$$

with the condition that this function satisfies $x_\alpha \in X_\alpha$. With this language, the "projection" $\prod_{\alpha\in A} X_\alpha \longrightarrow X_\alpha$ is simply the restriction along $\{\alpha\} \hookrightarrow \alpha$.

In the case that all X_α are the same set X, then $\prod_{\alpha\in A} X_\alpha$ is simply the set of functions $A \longrightarrow X$. So, the countably infinite product of $\mathbb{R}$ with itself is synonymous with the collection of sequences in $\mathbb{R}$.

Example 12.4. We mentioned above that the set of sequences in $\mathbb{R}$ is the infinite product $\prod_n \mathbb{R}$. What does a neighborhood of a sequence (x_n) look like in the product topology? We are only allowed to constrain finitely many coordinates, so a neighborhood consists of all sequences that are near to (x_n) in some fixed, finitely many coordinates.

Proposition 12.5. *Let $A_j \subseteq X_j$ for all $j \in J$. Then*

$$\prod_j \overline{A_j} = \overline{\prod_j A_j}$$

in both the product and box topologies.

Proof. As usual, we have two subsets of $\prod_j X_j$ we want to show are the same, so we establish that each is a subset of the other. The following proof works in both topologies under consideration.

($\subseteq$) Let $(x_j) \in \prod \overline{A_j}$. We use the neighborhood criterion of the closure to show that $(x_j) \in \overline{\prod_j A_j}$. Thus let $U = \prod_j U_j$ be a basic open neighborhood of (x_j). Then for each j, U_j is a neighborhood of x_j. Since $x_j \in \overline{A_j}$, it follows that U_j must meet A_j in some point, say y_j. It then follows that $(y_j) \in U \cap \prod_j A_j$. By the neighborhood criterion, it follows that $(x_j) \in \overline{\prod_j A_j}$.

($\supseteq$) Now suppose that $(x_j) \in \overline{\prod_j A_j}$. For each j, let U_j be a neighborhood of x_j. Then $p_j^{-1}(U_j)$ is a neighborhood of (x_j), so it must meet $\prod_j A_j$. But this means precisely that U_j meets A_j. It follows that $x_j \in \overline{A_j}$ for all j. ■

Note that this implies that an (arbitrary) product of closed sets is closed, using either the product or box topologies. In particular, I^2 is closed in $\mathbb{R}^2$ and T^2 is closed in $\mathbb{R}^4$.

13. Fri, Sept. 26

Proposition 13.1. *Suppose X_j is Hausdorff for each $j \in J$. Then so is $\prod_j X_j$ in both product and box topologies.*

Proof. Let $(x_j) \neq (x'_j) \in \prod_j X_j$. Then $x_\ell \neq x'_\ell$ for some particular ℓ. Since X_ℓ is Hausdorff, we can find disjoint neighborhoods U and U' of x_ℓ and x'_ℓ in X_ℓ. Then $p_\ell^{-1}(U)$ and $p_\ell^{-1}(U')$ are disjoint neighborhoods of (x_j) and (x'_j) in the product topology, so $\prod_j X_j$ is Hausdorff in the product topology.

For the box topology, we can either say that the above works just as well for the box topology, or we can say that since the box topology is a refinement of the product topology and the product topology is Hausdorff, it follows that the box topology must also be Hausdorff. ■

The converse is true as well. To see this, we use the fact that a subspace of a Hausdorff space is Hausdorff. How do we view X_ℓ as a subspace of $\prod_j X_j$? We can think about an axis inclusion. Thus pick $y_j \in X_j$ for $j \neq \ell$. We define

$$a_\ell : X_\ell \longrightarrow \prod_j X_j$$

by

$$a_\ell(x)_j = \begin{cases} x & j = \ell \\ y_j & j \neq \ell. \end{cases}$$

Note that, by the universal property of the product, in order to check that a_ℓ is continuous, it suffices to check that each coordinate map is continuous. But the coordinate maps are the identity and a lot of constant maps, all of which are certainly continuous. The map a_ℓ is certainly injective (assuming all X_j are nonempty!), and it is an example of an embedding.

Definition 13.2. A map $f : X \longrightarrow Y$ is said to be an **embedding** if it is a homeomorphism onto its image $f(X)$, equipped with the subspace topology.

We already discussed injectivity and continuity of the axis inclusion a_ℓ, so it only remains to show this is open, as a map to $a_\ell(X_\ell)$. Let $U \subseteq X_\ell$ be open. Then

$$a_\ell(U) = p_\ell^{-1}(U) \cap a_\ell(X_\ell),$$

so $a_\ell(U)$ is open in the subspace topology on $a_\ell(X_\ell)$.

We will often do the above sort of exercise: if we introduce a new property or construction, we will ask how well this interacts with other constructions/properties.

Here is another example of an embedding.

Example 13.3. Let $f : X \longrightarrow Y$ be continuous and define the graph of f to be

$$\Gamma(f) = \{(x, y) \mid y = f(x)\} \subseteq X \times Y.$$

The function

$$\gamma : X \longrightarrow X \times Y, \qquad \gamma(x) = (x, f(x))$$

is an embedding with image $\Gamma(f)$.

Let us verify that this is indeed an embedding. Injectivity is easy (this follows from the fact that one of the coordinate maps is injective), and continuity comes from the universal property for the product $X \times Y$ since id_X and f are both continuous. Note that $(p_Y)_{|\Gamma(f)}$, which is continuous since it is the restriction of the continuous projection p_Y, provides an inverse to γ.

What happens if we turn all of the arrows around in the defining property of a product? We might call such a thing a "coproduct". To be precise we would want a space that is universal among spaces equipped with maps *from* X and Y. In other words, given a space Z and maps $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$, we would want a unique map from the coproduct to Z, making the following diagram commute.

$$\begin{tikzcd}
X \arrow[dr] \arrow[drr, bend left, "f"] & & \\
& X \amalg Y \arrow[r, dashed, "\exists! h"] & Z \\
Y \arrow[ur] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

The glueing lemma gave us exactly such a description, in the case that our domain space X was made up of *disjoint* open subsets A and B. In general, the answer here is given by the **disjoint union**.

Recall that, as a set, the disjoint union of sets X and Y is the subset

$$X \amalg Y \subseteq (X \cup Y) \times \{1, 2\},$$

where $X \amalg Y = (X \times \{1\}) \cup (Y \times \{2\})$. More generally, given sets X_j for $j \in J$, their disjoint union $\coprod_j X_j$ is the subset

$$\coprod_j X_j \subseteq \left(\bigcup_j X_j \right) \times J$$

given by

$$\coprod_j X_j = \bigcup_j (X_j \times \{j\}).$$

There are natural inclusions $\iota_X : X \longrightarrow X \amalg Y$ or more generally $\iota_{X_j} : X_j \hookrightarrow \coprod_j X_j$. We topologize the coproduct by giving it the finest topology such that all ι_{X_j} are continuous. In other words, a subset $U \subseteq \coprod_j X_j$ is open if and only if $\iota_j^{-1}(U) \subseteq X_j$ is open for all j.

Note that in the case of a coproduct of two spaces, the subspace topology on $X \subseteq X \amalg Y$ agrees with the original topology on X. Furthermore, both X and Y are open in $X \amalg Y$, so the universal property for the coproduct is precisely the glueing lemma.