14. Monday, Sept. 29

On Friday, we introduced the idea of a coproduct, which is dual to the product. In the case of a space X which happens to be the union of two open, disjoint, subspaces A and B, then the glueing lemma told us that X satisfies the correct property to be the coproduct $X = A \amalg B$.

For a more general coproduct $\coprod_j X_j$, we declared $U \subseteq \coprod_j X_j$ to be open if and only if $\iota_j^{-1}(U)$ is open for all j. Let's verify that this satisfies the universal property.

Thus let $f_j : X_j \longrightarrow Z$ be continuous for all $j \in J$. It is clear that, set-theoretically, the various images $\iota_j(X_j)$ inside the coproduct are disjoint and that their union is the entire coproduct. So to define a function on the coproduct, it suffices to define a function on each $\iota_j(X_j)$. But each ι_j is injective, in other words a bijection onto its image, so defining $f_{|\iota_j(X_j)}$ is equivalent to defining $f_{|\iota_j(X_j)} \circ \iota_j$. But the latter, according to the universal property, is supposed to be f_j. So the upshot of all of this is that there is no choice in how we define the function f. As usual, we only need verify that this function f is continuous.

Let $V \subseteq Z$ be open. We wish to know that $f^{-1}(V)$ is open in $\coprod_j X_j$. But according to the topology on the coproduct, this amounts to showing that each $\iota_j^{-1} f^{-1}(V)$ is open. But this is $(f \circ \iota_j)^{-1}(V) = f_j^{-1}(V)$, which is open by the assumption that each f_j is continuous.

Example 14.1.
1. Consider $X = [0,1]$ and $Y = [2,3]$. Then in this case $X \amalg Y$ is homeomorphic to the subspace $X \cup Y$ of $\mathbb{R}$. The same is true of these two intervals are changed to be open or half-open.
2. Consider $X = (0,1)$ and $Y = \{1\}$. Then $X \amalg Y$ is **not** homeomorphic to $(0,1) \cup \{1\} = (0,1]$. The singleton $\{1\}$ is open in $X \amalg Y$ but not in $(0,1]$. Instead, $X \amalg Y$ is homeomorphic to $(0,1) \cup \{2\}$.
3. Similarly $(0,1) \amalg [1,2]$ is homeomorphic to $(0,1) \cup [2,3]$ but not to $(0,1) \cup [1,2] = (0,2]$.
4. In yet another similar example, $(0,2) \amalg (1,3)$ is homeomorphic to $(0,1) \cup (2,3)$ but not to $(0,2) \cup (1,3) = (0,3)$.

Proposition 14.2. *Let X_i be spaces, for $i \in I$. Then $\coprod_i X_i$ is Hausdorff if and only if all X_i are Hausdorff.*

Proof. This is even easier than for products. First, X_i always embeds as a subspace of the coproduct, so it follows that X_i is Hausdorff if the coproduct is as well. On the other hand, suppose all X_i are Hausdorff and suppose that $x \neq y$ are points of $\coprod_i X_i$. Either x and y come from different X_i's, in which case the X_i's themselves serve as the disjoint neighborhoods. The alternative is that x and y live in the same Hausdorff X_i, but then we can find disjoint neighborhoods in X_i. ■

The next important construction is that of a quotient, or identification space.

The general setup is that we have a surjective map $q : X \longrightarrow Y$, which we view as making an identification of points in X. More precisely, suppose that we have an equivalence relation $\sim$ on X. We can consider the set $X/\sim$ of equivalence classes in X. There is a natural surjective map $q : X \longrightarrow X/\sim$ which takes $x \in X$ to its equivalence class.

And in fact every surjective map is of this form. Suppose that $q : X \longrightarrow Y$ is surjective. We define a relation on X by saying that $x \sim x'$ if and only if $q(x) = q(x')$. Then the function $X/\sim \longrightarrow Y$ sending the class of x to $q(x)$ is a bijection.

We want to mimic the above situation in topology, but to understand what this should mean, we first look at the universal property of the quotient for sets. This says: if $f : X \longrightarrow Z$ is a function that is constant on the equivalence classes in X, then there is a (unique) factorization $g : X/\sim \longrightarrow Z$ with $g \circ q = f$.

We want to have a similar setup in topology. Said in the equivalence relation framework, given a space X and a relation $\sim$ on X, we want a continuous map $q : X \longrightarrow Y$ such that given any space Z with a continuous map $f : X \longrightarrow Z$ which is constant on equivalence classes, there is a unique continuous map $g : Y \longrightarrow Z$ such that $g \circ q = f$. By considering the cases in which Z is a set with the trivial topology, so that maps to Z are automatically continuous, we can see that on the level of sets $q : X \longrightarrow Y$ must be $X \longrightarrow X/\sim$. It remains only to specify the topology on $Y = X/\sim$.

We want the topological quotient to be the universal example of a continuous map out of X which is constant on equivalence classes. Universal here means that we always want to have a map $Y \longrightarrow Z$ whenever $f : X \longrightarrow Z$ is another such map. Since we want to construct maps *out of* Y, this suggests we should include as many open sets as possible in Y. This leads to the following definition.

Definition 14.3. We say that a surjective map $q : X \longrightarrow Y$ is a **quotient map** if $V \subseteq Y$ is open if and only if $q^{-1}(V)$ is open in X.

One implication is the definition of continuity, but the other is given by our desire to include as many opens as we can.

Proposition 14.4. *(Universal property of the quotient) Let $q : X \longrightarrow Y$ be a quotient map. If Z is any space, and $f : X \longrightarrow Z$ is any continuous map that is constant on the fibers[2] of q, then there exists a unique continuous $g : Y \longrightarrow Z$ such that $g \circ q = f$.*

Proof. It is clear how g must be defined: $g(y) = f(x)$ for any $x \in q^{-1}(y)$. It remains to show that g is continuous. Let $W \subseteq Z$ be open. We want $g^{-1}(W) \subseteq Y$ to be open as well. By the definition of a quotient map, $g^{-1}(W)$ is open if and only if $q^{-1}(g^{-1}(W)) = (g \circ q)^{-1}(W) = f^{-1}(W)$ is open, so we are done by continuity of f. ∎

Example 14.5. Define $q : \mathbb{R} \longrightarrow \{-1, 0, 1\}$ by

$$q(x) = \begin{cases} 0 & x = 0 \\ \frac{|x|}{x} & x \neq 0. \end{cases}$$

What is the resulting topology on $\{-1, 0, 1\}$? The points -1 and 1 are open, and the only open set containing 0 is the whole space.

Note that this example shows that a quotient of a Hausdorff space need not be Hausdorff.

Proposition 14.6. *Let $q : X \longrightarrow Y$ be a continuous, surjective, open map. Then q is a quotient map. The same is true if q is closed instead of open.*

Proof. One implication is simply the definition of continuity. For the other, suppose that $V \subseteq Y$ is a subset such that $q^{-1}(V) \subseteq X$ is open. Then $q(q^{-1}(V))$ is open since q is open. Finally, we have $V = q(q^{-1}(V))$ since q is surjective. ∎

The converse is not true, however, as the next example shows.

Example 14.7. Consider $q; \mathbb{R} \longrightarrow [0, \infty)$ given by

$$q(x) = \begin{cases} 0 & x \leq 0 \\ x & x \geq 0. \end{cases}$$

The quotient topology on $[0, \infty)$ is the same as the subspace topology it gets from $\mathbb{R}$. But this is not an open map, since the image of $(-2, -1)$ is $\{0\}$, which is not open.

[2]A "fiber" is simply the preimage of a point.

We discussed last time the fact that a quotient map need not be open. Nevertheless, there is a class of open sets that are always carried to open sets.

Definition 15.1. Let $q : X \longrightarrow Y$ be a continuous surjection. We say a subset $A \subseteq X$ is **saturated** (with respect to q) if it is of the form $q^{-1}(V)$ for some subset $V \subseteq Y$.

It follows that A is saturated if and only if $q^{-1}(q(A)) = A$. Recall that a **fiber** of a map $q : X \longrightarrow Y$ is the preimage of a single point. Then another description is that A is saturated if and only if it contains all fibers that it meets.

Proposition 15.2. *A continuous surjection $q : X \twoheadrightarrow Y$ is a quotient map if and only if it takes saturated open sets to saturated open sets.*

Proof. Exercise. ■

Last time, we defined the quotient topology coming from a continuous surjection $q : X \longrightarrow Y$. Recall that q is a quotient map (and Y has the quotient topology) if $V \subseteq Y$ is open precisely when $q^{-1}(V) \subseteq X$ is open.

Example 15.3. (Collapsing a subspace) Let $A \subseteq X$ be a subspace. We define a relation on X as follows: $x \sim y$ if both are points in A or if neither is in A and $x = y$. Here, we have one equivalence class for the subset A, and every point outside of A is its own equivalence class. Standard notation for the set $X/\sim$ of equivalence classes under this relation is X/A. The universal property can be summed up as saying that any map on X which is constant on A factors through the quotient X/A.

For example, we considered last time the example $\mathbb{R}/(-\infty, 0] \cong [0, \infty)$.

Example 15.4. Consider $\partial I \subseteq I$. The exponential map $e : I \longrightarrow S^1$ is constant on ∂I, so we get an induced continuous map $\varphi : I/\partial I \longrightarrow S^1$, which is easily seen to be a bijection. In fact, it is a homeomorphism. Once we learn about compactness, it will be easy to see that this is a closed map.

We show instead that it is open. A basis for $I/\partial I$ is given by $q(a, b)$ with $0 < a < b < 1$ and by $q([0, a) \cup (b, 1])$ with again $0 < a < b < 1$. It is clear that both are taken to basis elements for the subspace topology on S^1. It follows that φ is a homeomorphism.

Example 15.5. Generalizing the previous example, for any closed ball $D^n \subseteq \mathbb{R}^{n+1}$, we can consider the quotient $D^n/\partial D^n$. Exercise: define a surjective continuous map

$$q : D^n \longrightarrow S^n$$

taking the origin to the south pole and the boundary to the north pole. This then defines a continuous bijection $D^n/\partial D^n \longrightarrow S^n$, and we will see later in the course that this is automatically a homeomorphism.

Example 15.6. (Real projective space) On S^n we impose the equivalence relation $\mathbf{x} \sim -\mathbf{x}$. The resulting quotient space is known as n-dimensional real projective space and is denoted $\mathbb{RP}^n$.

Consider the case $n = 1$. We have the hemisphere inclusion $I \hookrightarrow S^1$ given by $x \mapsto e^{ix\pi}$. Then the composition $I \hookrightarrow S^1 \twoheadrightarrow \mathbb{RP}^1$ is a quotient map that simply identifies the boundary ∂I to a point. In other words, this is example **??** from above, and we conclude that $\mathbb{RP}^1 \cong S^1$. However, the higher-dimensional versions of these spaces are certainly not homeomorphic.

Example 15.7. (Complex projective space) Consider S^{2n-1} as a subspace of $\mathbb{C}^n$. We then have the coordinate-wise multiplication by elements of $S^1 \cong U(1)$ on $\mathbb{C}^n$. This multiplication restricts to a multiplication on the subspace S^{2n-1}, and we impose an equivalence relation $(z_1, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n)$ for all $\lambda \in S^1$. The resulting quotient space is the complex projective space $\mathbb{CP}^n$.

Example 15.8. (Torus) On $I \times I$, we impose the relation $(0, y) \sim (1, y)$ and also the relation $(x, 0) \sim (x, 1)$. The resulting quotient space is the torus $T^2 = S^1 \times S^1$. We recognize this as the product of two copies of example **??**, but beware that in general a product of quotient maps need not be a quotient map.

A number of the examples above have secretly been examples of a more general construction, namely the quotient under the action of a group.

Definition 15.9. A **topological group** is a based space (G, e) with a continuous multiplication $m : G \times G \longrightarrow G$ and inverse $i : G \longrightarrow G$ satisfying all of the usual axioms for a group.

Remark 15.10. Munkres requires all topological groups to satisfy the condition that points are closed. We will not make this restriction, though the examples we will consider will all satisfy this.

Example 15.11.
(1) Any group G can be considered as a topological group equipped with the discrete topology. For instance, we have the cyclic groups $\mathbb{Z}$ and $C_n = \mathbb{Z}/n\mathbb{Z}$.
(2) The real line $\mathbb{R}$ is a group under addition, This is a topological group because addition and multiplication by -1 are both continuous. Note that here $\mathbb{Z}$ is at the same time both a subspace and a subgroup. It is thus a topological subgroup.
(3) If we remove zero, we get the multiplicative group $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ of real numbers.
(4) Inside $\mathbb{R}^\times$, we have the subgroup $\{1, -1\}$ of order two.
(5) $\mathbb{R}^n$ is also a topological group under addition. In the case $n = 2$, we often think of this as $\mathbb{C}$.
(6) Again removing zero, we get the multiplicative group $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ of complex numbers.
(7) Inside $\mathbb{C}^\times$ we have the subgroup of complex numbers of norm 1, aka the circle group $S^1 \cong U(1) = SO(2)$.
(8) This last example suggests that matrix groups in general are good candidates. For instance, we have the topological group $Gl_n(\mathbb{R})$. This is a subspace of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. The determinant mapping $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is polynomial in the coefficients and therefore continuous. The general linear group is the complement of $\det^{-1}(0)$. It follows that $Gl_n(\mathbb{R})$ is an open subspace of $\mathbb{R}^{n^2}$.
(9) Inside $Gl_n(\mathbb{R})$, we have the closed subgroups $Sl_n(\mathbb{R})$, $O(n)$, $SO(n)$.

16. Fri, Oct. 3

Let G be a topological group and fix some $h \in G$. Define $L_h : G \longrightarrow G$ by $L_h(g) = hg$. This is left multiplication by h. The definition of topological group implies that this is continuous, as L_h is the composition

$$G \xrightarrow{(h,\mathrm{id})} G \times G \xrightarrow{m} G.$$

Moreover, $L_{h^{-1}}$ is clearly inverse to L_h and continuous by the same argument, so we conclude that each L_h is a homeomorphism. Since $L_h(e) = h$, we conclude that neighborhoods around h look like neighborhoods around e. Since h was arbitrary, we conclude that neighborhoods around one point look like neighborhoods around any other point. This implies that a space like the unoin of the coordinate axes in $\mathbb{R}^2$ cannot be given the structure of topological group, as neighborhoods around the origin do not resemble neighborhoods around other points.

The main reason for studying topological groups is to consider their *actions* on spaces.

Definition 16.1. Let G be a topological group and X a space. A **left action** of G on X is a map $a : G \times X \longrightarrow X$ which is associative and unital. This means that $a(g, a(h, x)) = a(gh, x)$ and

$a(e,x)=x$. Diagrammatically, this is encoded as the following commutative diagrams

$$\begin{tikzcd} G\times G\times X \arrow[r, "\mathrm{id}\times a"] \arrow[d, "m\times\mathrm{id}"'] & G\times X \arrow[d, "a"] \\ G\times X \arrow[r, "a"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "{e,\mathrm{id}}"] \arrow[dr, "\mathrm{id}"'] & G\times X \arrow[d, "a"] \\ & X. \end{tikzcd}$$

It is common to write $g\cdot x$ or simply gx rather than $a(g,x)$.

There is a similar notion of right action of G on X, given by a map $X\times G\longrightarrow X$ satisfying the appropriate properties.

Proposition 16.2. *Suppose that $(g,x)\mapsto g\cdot x$ is a left action of G on X. Then the assignment $(x,g)\mapsto g^{-1}\cdot x$ defines a right action of G on X.*

Proof. The only point of interest is the associativity property. We write $x\cdot g=g^{-1}\cdot x$. Then

$$(x\cdot g)\cdot h=h^{-1}\cdot(g^{-1}\cdot x)=(h^{-1}g^{-1})\cdot x=(gh)^{-1}\cdot x=x\cdot(gh),$$

which verifies that we have a right action. ■

Given an action of G on a space X, we define a relation on X by $x\sim y$ if $y=g\cdot x$ for some g. The equivalence classes are known as **orbits** of G in X, and the quotient of X by this relation is typically written as X/G. Really, the notation X/G should be reserved for the quotient by a *right action* of G on X, and the quotient by a left action should be $G\backslash X$.

Example 16.3. (1) For any G, left multiplication gives an action of G on itself! This is a transitive action, meaning that there is only one orbit, and the quotient $G\backslash G$ is just a point.

Note that we saw above that, for each $h\in G$, the map $L_h:G\longrightarrow G$ is a homeomorphism. This generalizes to any action. For each $g\in G$, the map $a(g,-):X\longrightarrow X$ is a homeomorphism.

(2) For any (topological) subgroup $H\le G$, left multiplication by elements of H gives a left action of H on G. Note that an orbit here is precisely a right coset Hg. The quotient is $H\backslash G$, the set of right cosets of H in G.

(3) The following example is interesting not for topological reasons but rather for the left action/ right action distinction. Let X be a space, n a natural number, and Σ_n the symmetric group on n letters. Then there is a natural action of Σ_n on X^n. In the literature, this is often described as a left action, but the simpler action that arises is a right action.

Note that Σ_n is the automorphism group (group of self-bijections) of the set $\mathbf{n}=\{1,\ldots,n\}$. We can regard X^n as the set of functions $\mathbf{n}\xrightarrow{x_{(-)}}X$. There is an obvious way to combine a bijection and a function, via composition. The assignment $(x,\sigma)\mapsto x\circ\sigma$ defines a right action of Σ_n on X^n.

As I mentioned, in the literature, there is frequent reference to a left action, but this is simply the left action $\sigma\cdot x_{(-)}:=x_{(-)}\cdot\sigma^{-1}$. In other words,

$$\sigma\cdot(x_1,\ldots,x_n)=(x_{\sigma^{-1}(1)},\ldots,x_{\sigma^{-1}(n)}).$$

(4) Consider the subgroup $\mathbb{Z}\le\mathbb{R}$. Since $\mathbb{R}$ is abelian, we don't need to worry about about left vs. right actions or left vs. right cosets. We then have the quotient $\mathbb{R}/\mathbb{Z}$, which is again a topological group (again, $\mathbb{R}$ is abelian, so $\mathbb{Z}$ is normal).

What is this group? Once again, consider the exponential map $\exp:\mathbb{R}\longrightarrow S^1$ given by $\exp(x)=e^{2\pi ix}$. This is surjective, and it is a homomorphism since

$$\exp(x+y)=\exp(x)\exp(y).$$

The First Isomorphism Theorem in group theory tells us that $S^1 \cong \mathbb{R}/\ker(\exp)$, at least as a group. The kernel is precisely $\mathbb{Z} \leq \mathbb{R}$, and it follows that $S^1 \cong \mathbb{R}/\mathbb{Z}$ as a group. To see that this is also a homeomorphism, we need to know that $\exp : \mathbb{R} \longrightarrow S^1$ is a quotient map, but this follows from our earlier verification that $I \longrightarrow S^1$ is a quotient. Another way to think about this is that the universal property of the quotient gives us continuous maps $I/\partial I \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow I/\partial I$ which are inverse to each other.