20. Mon, Oct. 13

What we have done so far corresponds roughly to Chapters 2 & 3 of Lee. Now we turn to Chapter 4.

The first idea is connectedness. Essentially, we want to say that a space cannot be decomposed into two disjoint pieces.

Definition 20.1. A **disconnection** (or separation) of a space X is a pair of disjoint, nonempty open subsets $U, V \subseteq X$ with $X = U \cup V$. We say that X is **connected** if it has no disconnection.

Example 20.2.
(1) If X is a discrete space (with at least two points), then any pair of disjoint nonempty subsets gives a disconnection of X.
(2) Let X be the subspace $(0,1) \cup (2,3)$ of $\mathbb{R}$. Then X is disconnected.
(3) More generally, if $X \cong A \coprod B$ for nonempty spaces A and B, then X is disconnected.
(4) Another example of a disconnected subspace of $\mathbb{R}$ is the subspace $\mathbb{Q}$. A disconnection of $\mathbb{Q}$ is given by $(-\infty, \pi) \cap \mathbb{Q}$ and $(\pi, \infty) \cap \mathbb{Q}$.
(5) Any set with the trivial topology is connected, since there is only one nonempty open set.
(6) Of the 29 topologies on $X = \{1, 2, 3\}$, 19 are connected, and the other 10 are disconnected. For example, the topology $\{\emptyset, \{1\}, X\}$ is connected, but $\{\emptyset, \{1\}, \{2, 3\}, X\}$ is not.
(7) If X is a space with the generic point (or included point) topology, in which the nonempty open sets are precisely the ones containing a special point x_0, then X is connected.
(8) If X is a space with the excluded point topology, in which the open proper subsets are the ones missing a special point x_0, then X is connected.
(9) The lower limit topology $\mathbb{R}_{\ell\ell}$ is disconnected, as the basis elements $[a, b)$ are both open and closed (clopen!), which means that their complements are open.

Proposition 20.3. *Let X be a space. The following are equivalent:*
(1) *X is disconnected*
(2) *$X \cong A \coprod B$ for nonempty spaces A and B*
(3) *There exists a nonempty, clopen, proper subset $U \subsetneq X$*
(4) *There exists a continuous surjection $X \twoheadrightarrow \{0, 1\}$, where $\{0, 1\}$ has the discrete topology.*

Now let's look at an interesting example of a connected space.

Proposition 20.4. *The only (nonempty) connected subspaces of $\mathbb{R}$ are singletons and intervals.*

Proof. It is clear that singletons are connected. Note that, by an interval, we mean simply a convex subset of $\mathbb{R}$. It is clear that any connected subset must be an interval since if A is connected and $a < b < c$ with $a, c \in A$, then either $b \in A$ or $(-\infty, b) \cap A$ and $(b, \infty) \cap A$ give a separation of A.

So it remains to show that intervals are connected. Let $I \subseteq \mathbb{R}$ be an interval with at least two points, and let $U \subseteq I$ be nonempty and clopen. We wish to show that $U = I$. Let $a \in U$. We will show that $U \cap [a, \infty) = I \cap [a, \infty)$. A similar argument will show that $U \cap (-\infty, a] = I \cap (-\infty, a]$.

Consider the set
$$R_a = \{b \in I \mid [a, b] \subseteq U\}.$$
Note that $a \in R_a$, so that R_a is nonempty. If R_a is not bounded above, then $[a, \infty) \subseteq U \subseteq I$, and we have our conclusion. Otherwise, the set R_a has a supremum $s = \sup R_a$ in $\mathbb{R}$. Since we can express s as a limit of a U-sequence and since U is closed in I, it follows that if $s \in I$ then s must also lie in U.

Note that if $s \notin I$, then since I is an interval we have
$$U \cap [a, \infty) = [a, s) = I \cap [a, \infty).$$
On the other hand, as we just said, if $s \in I$ then $s \in U$. But U is open, so some ϵ-neighborhood of s (in I) lies in U. But no point in $(s, s + \epsilon/2)$ can lie in U (or I), since any such point would then

also lie in R_a. Again, since I is an interval we have

$$U \cap [a, \infty) = [a, s] = I \cap [a, \infty).$$

■

One of the most useful results about connected spaces is the following.

Proposition 20.5. *Let $f : X \longrightarrow Y$ be continuous. If X is connected, then so is $f(X) \subseteq Y$.*

Proof. This is a one-liner. Suppose that $U \subseteq f(X)$ is closed and open. Then $f^{-1}(U)$ must be closed and open, so it must be either $\emptyset$ or X. This forces $U = \emptyset$ or $U = f(X)$. ■

Since the exponential map $\exp : [0, 1] \longrightarrow S^1$ is a continuous surjection, it follows that S^1 is connected. More generally, we have

Proposition 20.6. *Let $q : X \longrightarrow Y$ be a quotient map with X connected. Then Y is connected.*

Which of the other constructions we have seen preserve connectedness? All of them! (Well, except that subspaces of connected spaces need not be connected, as we have already seen.

Proposition 20.7. *Let $A_i \subseteq X$ be connected for each i, and assume that $x_0 \in \bigcap_i A_i \neq \emptyset$. Then $\bigcup_i A_i$ is connected.*

Proof. Assume each A_i is connected, and let $U \subseteq \bigcup_i A_i$ be nonempty and clopen. Then $x \in U$ for some $x \in \bigcup_i A_i$. Suppose $x \in A_{i_0}$. Then $U \cap A_{i_0}$ is nonempty and clopen in A_{i_0}, so $U \cap A_{i_0} = A_{i_0}$. In other words, $A_{i_0} \subseteq U$. Since $x_0 \in A_{i_0}$, it follows that $x_0 \in U$. But now for any other A_j, we have that $x_0 \in A_j \cap U$, so that $A_j \cap U$ is nonempty and clopen in A_j. It follows that $A_j \subseteq U$. ■

21. Wed, Oct. 15

Last time, we introduced the idea of connectedness and showed (1) that the connected subsets of $\mathbb{R}$ are precisely the intervals and (2) the image of a connected space under a continuous map is connected. This implies.

Theorem 21.1 (Intermediate Value Theorem). *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. Then f attains ever intermediate value between $f(a)$ and $f(b)$.*

Proof. This follows from the fact that the image is an interval. ■

We also showed that an overlapping union of connected subspaces is connected.
As an application, we get that products interact well with connectedness.

Proposition 21.2. *Assume $X_i \neq \emptyset$ for all $i \in \{1, \dots, n\}$. Then $\prod_{i=1}^{n} X_i$ is connected if and only if each X_i is connected.*

Proof. ($\Rightarrow$) This follows from Prop 20.5, as $p_i : \prod_i X_i \longrightarrow X_i$ is surjective (this uses that all X_j are nonempty).

($\Leftarrow$) Suppose each X_i is connected. By induction, it suffices to show that $X_1 \times X_2$ is connected. Pick any $z \in X_2$. We then have the embedding $X_1 \hookrightarrow X_1 \times X_2$ given by $x \mapsto (x, z)$. Since X_1 is connected, so is its image C in the product. Now for each $x_1 \in X_1$, we have an embedding $\iota_{x_1} : X_2 \hookrightarrow X_1 \times X_2$ given by $y \mapsto (x_1, y)$. Let $D_{x_1} = \iota_{x_1}(X_2) \cup C$. Note that each D is connected, being the overlapping union of two connected subsets. But we can write $X_1 \times X_2$ as the overlapping union of all of the D_{x_1}, so by the previous result the product is connected. ■

The following result is easy but useful.

Proposition 21.3. *Let $A \subseteq B \subseteq \overline{A}$ and suppose that A is connected. Then so is B.*

Proof. Exercise ■

Theorem 21.4. *Assume $X_i \neq \emptyset$ for all $i \in I$, where is I is arbitrary. Then $\prod_i X_i$ is connected if and only if each X_i is connected.*

Proof. As in the finite product case, it is immediate that if the product is connected, then so is each factor.

We sketch the other implication. We have already showed that each finite product is connected. Now let $(z_i) \in \prod_i X_i$. For each $j \in I$, write $D_j = p_j^{-1}(z_j) \subseteq \prod_i X_i$.

For each finite collection $j_1, \ldots, j_k \in I$, let

$$F_{j_1,\ldots,j_k} = \bigcap_{j \neq j_1,\ldots,j_k} D_j \subseteq \prod_i X_i.$$

Then $F_{j_1,\ldots,j_k} \cong X_{j_1} \times \cdots \times X_{j_k}$, so it follows that $F_{j_1,\ldots,j_k}$ is connected. Now $(z_i) \in F_{j_1,\ldots,j_k}$ for every such tuple, so it follows that

$$F = \bigcup F_{j_1,\ldots,j_k}$$

is connected.

It remains to show that F is dense in $\prod_i X_i$ (in other words, the closure of F is the whole product). Let

$$U = p_{j_1}^{-1}(U_{j_1}) \cap \cdots \cap p_{j_k}^{-1}(U_{j_k})$$

be a nonempty basis element. Then U meets $f_{j_1,\ldots,j_k}$, so U meets F. Since U was arbitrary, it follows that F must be dense. ■

Note that the above proof would not have worked with the box topology. We can show directly that $\mathbb{R}^{\mathbb{N}}$, equipped with the box topology, is not connected. Consider the subset $\mathcal{B} \subset \mathbb{R}^{\mathbb{N}}$ consisting of bounded sequences. If $(z_i) \in \mathcal{B}$, then $\prod_i (z_i - 1, z_i + 1)$ is a neighborhood of (z_i) in $\mathcal{B}$. On the other hand, if $(z_i) \notin \mathcal{B}$, the same formula gives a neighborhood consisting entirely of unbounded sequences. We conclude that $\mathcal{B}$ is a nontrivial clopen set in the box topology.

Ok, so we have looked at examples and studied this notion of being connected, but if you asked your calculus students to describe what it should mean for a subset of $\mathbb{R}$ to be connected, they probably wouldn't come up with the "no nontrivial clopen subsets" idea. Instead, they would probably say something about being able to connect-the-dots. In other words, you should be able to draw a line from one point to another while staying in the subset. This leads to the following idea.

Definition 21.5. We say that $A \subseteq X$ is **path-connected** if for every pair a, b of points in A, there is a continuous function (a path) $\gamma : I \longrightarrow A$ with $\gamma(0) = a$ and $\gamma(1) = b$.

This is not unrelated to the earlier notion.

Proposition 21.6. *If $A \subseteq X$ is path-connected, then it is also connected.*

Proof. Pick a point $a_0 \in A$. For any other $b \in A$, we have a path γ_b in A from a_0 to b. Then the image $\gamma_b(I)$ is a connected subset of A containing both a_0 and b. It follows that

$$A = \bigcup_{b \in A} \gamma_b(I)$$

is connected, as it is the overlapping union of connected sets. ∎

For subsets $A \subseteq \mathbb{R}$, we have

$$A \text{ is path-connected} \Rightarrow A \text{ is connected} \Leftrightarrow A \text{ is an interval} \Rightarrow A \text{ is path-connected.}$$

So the two notions coincide for subsets of $\mathbb{R}$. But the same is not true in $\mathbb{R}^2$!

Example 21.7 (Topologist's sine curve)**.** Let Γ be the graph of $\sin(1/x)$ for $x \in (0, \pi]$.Then Γ is homeomorphic to $(0, \pi]$ and is therefore path-connected and connected. Let C be the closure of Γ in $\mathbb{R}^2$. Then C is connected, as it is the closure of a connected subset. However, it is not path-connected (HW VI), as there is no path in C connecting the origin to the right end-point $(\pi, 0)$.

22. Fri, Oct. 17

Path-connectedness has much the same behavior as connectedness.

Proposition 22.1.

(1) *Images of path-connected spaces are path-connected*
(2) *Overlapping unions of path-connected spaces are path-connected*
(3) *Finite products of path-connected spaces are path-connected*

However, the topologist's sine curve shows that closures of path-connected subsets need not be path-connected.

Our proof of connectivity of $\prod_i X_i$ last time used this closure property for connected sets, so the earlier argument does not adapt easily to path-connectedness. But it turns out to be easier to prove.

Theorem 22.2. *Assume $X_i \neq \emptyset$ for all $i \in I$, where is I is arbitrary. Then $\prod_i X_i$ is path-connected if and only if each X_i is path-connected.*

Proof. The interesting direction is $(\Leftarrow)$. Thus assume that each X_i is path-connected. Let (x_i) and (y_i) be points in the product $\prod_i X_i$. Then for each $i \in \mathcal{I}$ there is a path γ_i in X_i with $\gamma_i(0) = x_i$ and $\gamma_i(1) = y_i$. By the universal property of the product, we get a continuous path

$$\gamma = (\gamma_i) : [0,1] \longrightarrow \prod_i X_i$$

with $\gamma(0) = (x_i)$ and $\gamma(1) = (y_i)$. ∎

The overlapping union property for (path-)connectedness allows us to make the following definition.

Definition 22.3. Let $x \in X$. We define the **connected component** (or simply component) of x in X to be

$$C_x = \bigcup_{\substack{x \in C \\ \text{connected}}} C.$$

Similarly, the **path-component** of X is defined to be

$$PC_x = \bigcup_{\substack{x \in P \\ \text{connected}}} P.$$

The overlapping union property guarantees that C_x is connected and that PC_x is path-connected. Since path-connected sets are connected, it follows that for any x, we have $PC_x \subseteq C_x$. An immediate consequence of the above definition(s) is that any (path-)connected subset of X is contained in some (path-)component.

Example 22.4. Consider $\mathbb{Q}$, equipped with the subspace topology from $\mathbb{R}$. Then the only connected subsets are the singletons, so $C_x = \{x\}$. Such a space is said to be **totally disconnected**.

Note that for any space X, each component C_x is closed as $\overline{C_x}$ is a connected subset containing x, which implies $\overline{C_x} \subseteq C_x$. If X has finitely many components, then each component is the complement of the finite union of the remaining components, so each component is also open, and X decomposes as a disjoint union

$$X \cong C_1 \amalg C_2 \amalg \cdots \amalg C_n$$

of its components. But this does not happen in general, as the previous example shows.

The situation is worse for path-components: they need not be open or closed, as the topologist's sine curve shows.

Definition 22.5. Let X be a space. We say that X is **locally connected** if any neighborhood U of any point x contains a connected neighborhood $x \in V \subset U$. Similarly X is **locally path-connected** if any neighborhood U of any point x contains a path-connected neighborhood $x \in V \subset U$.

This may seem like a strange definition, but it has the following nice consequence.

Proposition 22.6. *Let X be a space. The following are equivalent.*

1. *X is locally connected*
2. *X has a basis consisting of connected open sets*
3. *for every open set $U \subseteq X$, the components of U are open in X*

Proof. We show (1) $\Leftrightarrow$ (3).

Suppose that X is locally connected and let $U \subseteq X$ be open. Take $C \subseteq U$ to be a component. Let $x \in C$. We can then find a connected neighborhood $x \in V \subseteq U$. Since C is the component of x, we must have $V \subseteq C$, which shows that C is open.

Suppose, on the other hand, that (3) holds. Let U be a neighborhood of x. Then the component C_x of x in U is the desired neighborhood V. ■

In particular, this says that the components are open if X is locally connected.

The locally path-connected property is even better.

Proposition 22.7. *Let X be a space. The following are equivalent.*

1. *X is locally path-connected*
2. *X has a basis consisting of path-connected open sets*
3. *for every open set $U \subseteq X$, the path-components of U are open in X*
4. *for every open set $U \subseteq X$, every component of U is path-connected and open in X.*

Proof. The implications (1) $\Leftrightarrow$ (3) are similar to the above. We argue for (1) $\Leftrightarrow$ (4).

Assume X is locally path-connected, and let C be a component of an open subset $U \subseteq X$. Let $P \subseteq C$ be a nonempty path-component. Then P is open in X. But all of the other path-components of C are also open, so their union, which is the complement of P, must be open. It follows that P is closed. Since C is connected, we must have $P = C$.

On the other hand, suppose that (4) holds. Let U be a neighborhood of x. Then the component C_x of x in U is the desired neighborhood V. ■

In particular, this says that the components and path-components agree if X is locally path-connected.

Just as path-connected implies connected, locally path-connected implies locally-connected. But, unfortunately, there are no other implications between the four properties.

Example 22.8. The topologist's sine curve is connected, but not path-connected or locally connected or locally path-connected (see HWVI). Thus it is possible to be connected but not locally so.

Example 22.9. For any space X, the **cone** on X is defined to be $CX = X \times [0,1]/X \times \{1\}$. The cone on any space is always path-connected. In particular, the cone on the topologist's sine curve is connected and-path connected but not locally connected or locally path-connected.

Example 22.10. A disjoint union of two topologist's sine curves gives an example that is not connected in any of the four ways.

Example 22.11. Note that if X is locally path-connected, then connectedness is equivalent to path-connectedness. A connected example would be $\mathbb{R}$ or a one-point space. A disconnected example would be $(0,1) \cup (2,3)$ or a two point (discrete) space.

Finally, we have spaces that are locally connected but not locally path-connected.

Example 22.12. The cocountable topology on $\mathbb{R}$ is connected and locally connected but not path-connected or locally path-connected. (See HWVI)

Example 22.13. The cone on the cocountable topology will give a connected, path-connected, locally connected space that is not locally path-connected.

Example 22.14. Two copies of $\mathbb{R}_{\text{cocountable}}$ give a space that is locally connected but not connected in the other three ways.