23. Mon, Oct. 20

The next topic is one of the major ones in the course: compactness. As we will see, this is the analogue of a "closed and bounded subset" in a general space. The definition relies on the idea of coverings.

Definition 23.1. An **open cover** of X is a collection $\mathcal{U}$ of open subsets that cover X. In other words, $\bigcup_{U\in\mathcal{U}} U = X$. Given two covers $\mathcal{U}$ and $\mathcal{V}$ of X, we say that $\mathcal{V}$ is a **subcover** if $\mathcal{V} \subseteq \mathcal{U}$.

Definition 23.2. A space X is said to be **compact** if every open cover has a *finite* subcover (i.e. a cover involving finitely many open sets).

Example 23.3. Clearly any finite topological space is compact, no matter the topology.

Example 23.4. An infinite set with the discrete topology is *not* compact, as the collection of singletons gives an open cover with no finite subcover.

Example 23.5. $\mathbb{R}$ is not compact, as the open cover $\mathcal{U} = \{(-n, n) \mid n \in \mathbb{N}\}$ has no finite subcover.

Example 23.6. Similarly $[0,\infty)$ is not compact, as the open cover $\mathcal{U} = \{[0, n)\}$ has no finite subcover. Recall that $[0,\infty) \cong [a, b)$.

Theorem 23.7. *Let $a < b$. Then $[a, b]$ is a compact subset of $\mathbb{R}$.*

Proof. Let $\mathcal{U}$ be an open cover. Then some element of the cover must contain a. Pick such an element and call it U_1.

Consider the set

$$\mathcal{E} = \{c \in [a, b] \mid [a, c] \text{ is finitely covered by } \mathcal{U}\}.$$

Certainly $a \in \mathcal{E}$ and $\mathcal{E}$ is bounded above by b. By the Least Upper Bound Axiom, $s = \sup \mathcal{E}$ exists. Note that $a \le s \le b$, so we must have $s \in U_s$ for some $U_s \in \mathcal{U}$. But then for any $c < s$ with $c \in U_s$, we have $c \in \mathcal{E}$. This means that

$$[a, c] \subseteq U_1 \cup \cdots \cup U_k$$

for $U_1, \ldots, U_k \in \mathcal{U}$. But then $[a, s] \subseteq U_1 \cup \cdots \cup U_k \cup U_s$. This shows that $s \in \mathcal{E}$. On the other hand, the same argument shows that for any $s < d < b$ with $d \in U_s$, we would similarly have $d \in \mathcal{E}$. Since $s = \sup \mathcal{E}$, there cannot exist such a d. This implies that $s = b$. ∎

Like connectedness, compactness is preserved by continuous functions.

Proposition 23.8. *Let $f : X \longrightarrow Y$ be continuous, and assume that X is compact. Then $f(X)$ is compact.*

Proof. Let $\mathcal{V}$ be an open cover of $f(X)$. Then $\mathcal{U} = \{f^{-1}(V) \mid V \in \mathcal{V}\}$ is an open cover of X. Let $\{U_1, \ldots, U_k\}$ be a finite subcover. It follows that the corresponding $\{V_1, \ldots, V_k\}$ is a finite subcover of $\mathcal{V}$. ∎

Example 23.9. Recall that we have the quotient map $\exp : [0, 1] \longrightarrow S^1$. It follows that S^1 is compact.

Theorem 23.10 (Extreme Value Theorem)**.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. Then f attains a maximum and a minimum.*

Proof. Since f is continuous and $[a, b]$ is both connected and compact, the same must be true of its image. But the compact, connected subsets are precisely the closed intervals. ∎

The following result is also quite useful.

Proposition 23.11. *Let X be Hausdorff and let $A \subseteq X$ be a compact subset. Then A is closed in X.*

Proof. Pick any point $x \in X \setminus A$ (if we can't, then $A = X$ and we are done). For each $a \in A$, we have disjoint neighborhoods $a \in U_a$ and $x \in V_a$. Since the U_a cover A, we only need finitely many, say $U_{a_1}, \dots, U_{a_k}$ to cover A. But then the intersection

$$V = V_{a_1} \cap \dots \cap V_{a_k}$$

of the corresponding V_a's is disjoint from the union of the U_a's and therefore also from A. Since V is a finite intersection of open sets, it is open and thus gives a neighborhood of x in $X \setminus A$. It follows that A is closed. ∎

Exercise 23.12. *If $A \subseteq X$ is closed and X is compact, then A is compact.*

Combining these results gives the following long-awaited consequence.

Corollary 23.13. *Let $f : X \longrightarrow Y$ be continuous, where X is compact and Y is Hausdorff, then f is a closed map.*

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In particular, if f is already known to be a continuous bijection, then it is automatically a homeomorphism. For example, this shows that the map $I/\partial I \longrightarrow S^1$ is a homeomorphism. Similarly, from Example 15.5 we have $D^n/\partial D^n \cong S^n$.

We will next show that finite products of compact spaces are compact, but we first need a lemma.

Lemma 24.1 (Tube Lemma)**.** *Let X be compact and Y be any space. If $W \subseteq X \times Y$ is open and contains $X \times \{y\}$, then there is a neighborhood V of y with $X \times V \subseteq W$.*

Proof. For each $x \in X$, we can find a basic neighborhood $U_x \times V_x$ of (x, y) in W. The U_x's give an open cover of X, so we only need finitely many of them, say $U_{x_1}, \dots, U_{x_n}$. Then we may take $V = V_{x_1} \cap \dots \cap V_{x_n}$. ∎

Proposition 24.2. *Let X and Y be nonempty. Then $X \times Y$ is compact if and only if X and Y are compact.*

Proof. As for connectedness, the continuous projections make X and Y compact if $X \times Y$ is compact.

Now suppose that X and Y are compact and let $\mathcal{U}$ be an open cover. For each $y \in Y$, the cover $\mathcal{U}$ of $X \times Y$ certainly covers the slice $X \times \{y\}$. This slice is homeomorphic to X and therefore finitely-covered by some $\mathcal{V} \subset \mathcal{U}$. By the Tube Lemma, there is a neighborhood V_y of y such that the tube $X \times V_y$ is covered by the same $\mathcal{V}$. Now the V_y's cover Y, so we only need finitely many of these to cover X. Since each tube is finitely covered by $\mathcal{U}$ and we can cover $X \times Y$ by finitely many tubes, it follows that $\mathcal{U}$ has a finite subcover. ∎

Theorem 24.3 (Heine-Borel)**.** *A subset $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded (contained in a single metric ball).*

Proof. Suppose A is compact. Then A must be closed in $\mathbb{R}^n$ since $\mathbb{R}^n$ is Hausdorff. To see that A is bounded, pick any point $a \in A$ (if A is empty, we are certainly done). Then the collection of balls $B_n(a) \cap A$ gives an open cover of A, since any other point in A is a finite distance away from a. Since A is compact, there must be a finite subcover $\{B_{n_1}(a), \dots, B_{n_k}(a)\}$. Let $N = \max\{n_1, \dots, n_k\}$. Then $A \subseteq B_N(a)$.

On the other hand, suppose that A is closed and bounded in $\mathbb{R}^n$. Since A is bounded, it is contained in $[-k, k]^n$ for some $k > 0$. But this product of intervals is compact since each interval is compact. Now A is a closed subset of a compact space, so it is compact. ∎

In fact, the forward implication of the above proof works to show that

Proposition 24.4. *Let $A \subseteq X$, where X is metric and A is compact. Then A is closed and bounded in X.*

But the reverse implication is not true in general, as the next example shows.

Example 24.5. Consider $[0,1] \cap \mathbb{Q} \subseteq \mathbb{Q}$. This is certainly closed and bounded, but we will see it is not compact. Consider the open cover $\mathcal{U} = \left\{[0, \frac{1}{\pi} - \frac{1}{n})\right\}_{n\in\mathbb{N}} \cup \{(\frac{1}{\pi}, 1]\}$.

Again, we have shown that compactness interacts well with finite products, and we would like a similar result in the arbitrary product case. This is a major theorem, known as the Tychonoff theorem. First, we show the theorem does not hold with the box topology.

Example 24.6. Let $D = [-1,1]$ and consider $D^{\mathbb{N}}$, equipped with the box topology. For each k, let introduce $L = [0,1)$ and $R = (0,1]$. Take cover by products of L's and R's. No finite subcover.

Theorem 24.7 (Tychonoff)**.** *Let $X_i \neq \emptyset$ for all $i \in \mathcal{I}$. Then $\prod_i X_i$ is compact if and only if each X_i is compact.*

We will prove this next time. Our proof, even for the difficult direction, will use the axiom of choice. In fact, Tychonoff's theorem is equivalent to the axiom of choice.

Theorem 24.8. *Tychonoff $\Rightarrow$ axiom of choice.*

Proof. This argument is quite a bit simplier than the other implication. Let $X_i \neq \emptyset$ for all $i \in \mathcal{I}$. We want to show that $X = \prod_i X_i \neq \emptyset$.

For each i, define $Y_i = X_i \cup \{\infty_i\}$, where $\infty_i \notin X_i$. We topologize Y_i such that the only nontrivial open sets are X_i and $\{\infty_i\}$. Now for each i, let $U_i = p_i^{-1}(\infty_i)$. The collection $\mathcal{U} = \{U_i\}$ gives a collection of open subsets of $Y = \prod_i Y_i$, and this collection covers Y if and only if $X = \emptyset$. Each Y_i is compact since it has only four open sets. Thus Y must be compact by the Tychonoff theorem. But no finite subcollection of $\mathcal{U}$ can cover Y. For example, $U_i \cup U_j$ does not cover Y since if $a \in X_i$ and $b \in X_j$, then we can define $(y_i) \in Y \setminus (U_i \cup U_j)$ by

$$y_k = \begin{cases} a & k = i \\ b & k = j \\ \infty_k & k \neq i, j \end{cases}$$

The same kind of argument will work for any finite collection of U_i's. Since $\mathcal{U}$ has no finite subcover and Y is compact, $\mathcal{U}$ cannot cover Y, so that X must be nonempty. ■

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Started by correcting Example 24.6.

Here is a simpler example of a noncompact product in the box topology. Consider $\{0,1\}^{\mathbb{N}}$. In the box topology, this space is discrete. Since it is infinite, it is not compact.

It turns out that the Tychonoff Theorem is *equivalent* to the axiom of choice. We will thus use a form of the axiom of choice in order to prove it.

Zorn's Lemma. Let P be a partially-ordered set. If every linearly-ordered subset of P has an upper bound in P, then P contains at least one maximal element.

Theorem 25.1 (Tychonoff)**.** *Let $X_i \neq \emptyset$ for all $i \in \mathcal{I}$. Then $\prod_i X_i$ is compact if and only if each X_i is compact.*

Proof. As we have seen a number of times, the implication ($\Rightarrow$) is trivial.

We now show the contrapositive of ($\Leftarrow$). Thus assume that $X = \prod_i X_i$ is not compact. We wish to conclude that one of the X_i must be noncompact. By hypothesis, there exists an open cover $\mathcal{U}$ of X with no finite subcover.

We first deal with the following case.

Special case: *$\mathcal{U}$ is a cover by prebasis elements.*
For each $i \in \mathcal{I}$, let $\mathcal{U}_i$ be the collection

$$\mathcal{U}_i = \{V \subseteq X_i \text{ open } \mid p_i^{-1}(V) \in \mathcal{U}\}.$$

For some i, the collection $\mathcal{U}_i$ must cover X_i, since otherwise we could pick $x_i \in X_i$ for each i with x_i not in the union of $\mathcal{U}_i$. Then the element $(x_i) \in \prod_i X_i$ would not be in $\mathcal{U}$ since it cannot be in any $p_i^{-1}(V)$. Then $\mathcal{U}_i$ cannot have a finite subcover, since that would provide a corresponding subcover of $\mathcal{U}$. It follows that X_i is not compact.

It remains to show that we can always reduce to the cover-by-prebasis case.

Consider the collection $\mathcal{N}$ of open covers of X having no finite subcovers. By assumption, this set is nonempty, and we can partially order $\mathcal{N}$ by inclusion of covers. Furthermore, if $\{\mathcal{U}_\alpha\}$ is a linearly order subset of $\mathcal{N}$, then $\mathcal{U} = \bigcup_\alpha \mathcal{U}_\alpha$ is an open cover, and it cannot have a finite subcover since a finite subcover of $\mathcal{U}$ would be a finite subcover of one of the $\mathcal{U}_\alpha$. Thus $\mathcal{U}$ is an upper bound in $\mathcal{N}$ for $\{\mathcal{U}_\alpha\}$. By Zorn's Lemma, $\mathcal{N}$ has a maximal element $\mathcal{V}$.

Now let $\mathcal{S} \subseteq \mathcal{V}$ be the subcollection consisting of the prebasis elements in $\mathcal{V}$. We claim that $\mathcal{S}$ covers X. *Suppose not.* Thus let $x \in X$ such that x is not covered by $\mathcal{S}$. Then x must be in V for some $V \in \mathcal{V}$. By the definition of the product topology, x must have a basic open neighborhood in $B \subset V$. But any basic open set is a finite intersection of prebasic open sets, so $B = S_1 \cap \dots S_k$. If x is not covered by $\mathcal{S}$, then none of the S_i are in $\mathcal{S}$. Thus $\mathcal{V} \cup \{S_i\}$ is not in $\mathcal{N}$ by maximality of $\mathcal{V}$. In other words, $\mathcal{V} \cup \{S_i\}$ has a finite subcover $\{V_{i,1}, \dots, V_{i,n_i}, S_i\}$. Let us write

$$\hat{V}_i = V_{i,1} \cup \dots \cup V_{i,n_i}.$$

Now

$$X = \bigcap_i \left(S_i \cup \hat{V}_i\right) \subseteq \left(\bigcap_i S_i\right) \cup \left(\bigcup_i \hat{V}_i\right) \subseteq V \cup \left(\bigcup_i \hat{V}_i\right)$$

This shows that $\mathcal{V}$ has a finite subcover, which contradicts that $\mathcal{V} \in \mathcal{N}$. We thus conclude that $\mathcal{S}$ covers X using only prebasis elements.

But now by the argument at the beginning of the proof, $\mathcal{S}$, and therefore $\mathcal{V}$ as well, has a finite subcover. This is a contradiction. ∎