Math 654 - Algebraic Topology
Homework 11
Fall 2015

1. Let
$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$
be a short exact sequence and D any abelian group.
   (a) Show that the following sequence is exact:
$$0 \longrightarrow \mathrm{Hom}(D, A) \longrightarrow \mathrm{Hom}(D, B) \longrightarrow \mathrm{Hom}(D, C).$$
   (b) Show that the following sequence is exact:
$$0 \longrightarrow \mathrm{Hom}(C, D) \longrightarrow \mathrm{Hom}(B, D) \longrightarrow \mathrm{Hom}(A, D).$$

2. (Ext and extensions). An **extension** of A by M is a short exact sequence
$$0 \longrightarrow M \longrightarrow E \longrightarrow A \longrightarrow 0.$$
An equivalence of extensions $E \sim E'$ is a homomorphism $E \xrightarrow{\rho} E'$ making
$$\begin{array}{ccccccc} & & & E & & & \\ & & \nearrow & \downarrow{\scriptstyle\rho} & \searrow & & \\ 0 \longrightarrow & M & & & & A & \longrightarrow 0 \\ & & \searrow & & \nearrow & & \\ & & & E' & & & \end{array}$$
commute. (Note that ρ is automatically an isomorphism by the 5-lemma.) Denote by $\mathrm{Exten}(A, M)$ the set of equivalence classes of extensions of A by M.
   (a) Construct a function $\Phi : \mathrm{Ext}(A, M) \longrightarrow \mathrm{Exten}(A, M)$ as follows. Starting from a class $\alpha \in \mathrm{Ext}(A, M)$ and a resolution F_* of A, pick a representative $F_1 \xrightarrow{f} M$ of α. Then consider the diagram
$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & {\scriptstyle f_1}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & M \oplus_{F_1} F_0 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$
   (b) Construct a function $\Lambda : \mathrm{Exten}(A, M) \longrightarrow \mathrm{Ext}(A, M)$ as follows. Given an extension $M \longrightarrow E \longrightarrow A$, consider the 6-term exact sequence arising from the functor $\mathrm{Hom}(-, M)$.
   (c) Show that Φ and Λ are inverse to each other.

3. An abelian group M is said to be p-divisible if $M \xrightarrow{p} M$ (multiplication by p) is surjective. What does problem 2 tell you about extensions of $\mathbb{Z}/p$ by a p-divisible abelian group M?