Math 654 - Algebraic Topology
Homework 12
Fall 2015

1. In this problem, A denotes a finitely generated abelian group. Given E and E' in $\mathrm{Exten}(A, M)$, define P to be the pullback $P = \{(e, e') \mid q_E(e) = q_{E'}(e')\}$.

$$\begin{array}{ccc} P & \longrightarrow & E' \\ \downarrow & & \downarrow{\scriptstyle q_{E'}} \\ E & \underset{q_E}{\longrightarrow} & A. \end{array}$$

Let $j : M \hookrightarrow P$ be $j(m) = (i_E(m), -i_{E'}(m))$, and define $Q := \mathrm{coker}(M \xrightarrow{j} P)$.

(a) Let $i_Q : M \longrightarrow Q$ be given by $i_Q(m) = (m, 0)$. Show that

$$0 \longrightarrow M \xrightarrow{i_Q} Q \longrightarrow A \longrightarrow 0$$

is an extension. This is called the **Baer sum** of E and E'.

(b) Show that the operation of Baer sum makes $\mathrm{Exten}(A, M)$ into a group, where the split extension is the identity element.

(c) Show that either Φ or Λ from problem 11.2 is a homomorphism.

2. Let X be a Moore space of type $M(\mathbb{Z}/k\mathbb{Z}, n)$. Show that the quotient map

$$q : X \longrightarrow X/X^n \cong S^{n+1}$$

induces the trivial map on $\widetilde{\mathrm{H}}_i(-;\mathbb{Z})$ for all i, but not on $\mathrm{H}^{n+1}(-;\mathbb{Z})$. Deduce that the splitting in the Universal Coefficient Theorem cannot be natural.

3. Let $\mathbb{F}$ be a field. Give an identification of $\mathrm{H}^n(X;\mathbb{F})$ as the dual vector space of $\mathrm{H}_n(X;\mathbb{F})$.