Math 654 - Algebraic Topology
Homework 13
Fall 2015

1. In this problem, cohomology means singular cohomology, and $C^*(X)$ means singular cochains. Note that $C_*(X)$ is the free abelian group on the set of paths in X, so that $C^*(X; M)$ is the group of M-valued functions on the set of paths in X.

   (a) Show that if $\varphi \in C^1(X; M)$ is a cocyle, then $\varphi(g * f) = \varphi(g) + \varphi(f)$, where $g * f$ is path-composition. (Assume f and g are composable paths.)

   (b) Show that $\varphi(f) = \varphi(g)$ if $f \simeq_p g$.

   (c) Show that the above gives a well-defined homomorphism $\mathrm{H}^1(X; M) \longrightarrow \mathrm{Hom}(\pi_1(X), M)$. Explain why the Universal Coefficients theorem tells you these groups are isomorphic if X is path-connected.

2. Recall that $\mathbb{RP}^2$ is a Moore space of type $M(\mathbb{Z}/2\mathbb{Z}, 1)$. We showed in class that

$$\mathrm{H}^*(\mathbb{RP}^2; \mathbf{F}_2) \cong \mathbf{F}_2[x_1]/x_1^3.$$

   Show, on the other hand, that if p is odd then for $X = M(\mathbb{Z}/p\mathbb{Z}, 1)$, we have $x_1^2 = 0$, where x_1 is any class in $\mathrm{H}^1(X; \mathbf{F}_p) \cong \mathbf{F}_p$. (Hint: Use (graded)-commutativity.)

3. Compute the cohomology ring $\mathrm{H}^*(K; \mathbf{F}_2)$, where K is the Klein bottle, using cup products in simplicial cohomology.