Math 654 - Algebraic Topology
Homework 5
Fall 2015

1. A chain map that induces an isomorphism in homology is called a **quasi-isomorphism**. We showed in class that any chain homotopy equivalence is a quasi-isomorphism. Give an example of a quasi-isomorphism of chain complexes which is not a chain homotopy equivalance.

2. Show that the chain complex $C_*^{\Delta}(\mathbb{RP}^2)$ described in class (on 9-9-15) is chain homotopy equivalent to the complex $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$.

3. An **additive functor** $F : \mathbf{AbGp} \longrightarrow \mathbf{AbGp}$ is a functor such that each function

$$F : \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(FA, FB)$$

is a homomorphism. Show that if F is additive, then

$$F(0) = 0 \qquad \text{and} \qquad F(A \oplus B) \cong F(A) \oplus F(B).$$

4. Recall that a **short exact sequence** is a sequence of homomorphisms

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

which is exact (has trivial homology) at each spot.

   (a) A short exact sequence is called **split exact** if $B \cong A \oplus C$. Show that the following are equivalent for the (solid arrow) sequence:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, shift left, "i"] & B \arrow[l, dashed, shift left, "r"] \arrow[r, shift left, "p"] & C \arrow[l, dashed, shift left, "s"] \arrow[r] & 0 \end{tikzcd}$$

      i. The sequence is split exact
      ii. There exists a map s such that $p \circ s = \mathrm{id}_C$
      iii. There exists a map r such that $r \circ i = \mathrm{id}_A$

   (b) Consider the functor $\mathbb{Z}/2\mathbb{Z} \otimes - : \mathbf{AbGp} \longrightarrow \mathbf{AbGp}$. By applying this to the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

   show that this functor does not preserve short exact sequences.

   (c) Show, on the other hand, that *any* additive functor $F : \mathbf{AbGp} \longrightarrow \mathbf{AbGp}$ must preserve split exact sequences.