Math 654 - Algebraic Topology
Homework 8
Fall 2016

1. Given based spaces (X, x_0) and (Y, y_0), there is a natural axes inclusion $X \vee Y \hookrightarrow X \times Y$. Define the **smash product** of X and Y to be

$$X \wedge Y = (X \times Y)/(X \vee Y).$$

Show that there is a homeomorphism $S^1 \wedge S^1 \cong S^2$ or that more generally $S^n \wedge S^k \cong S^{n+k}$. (Hint: Feel free to assume the existence of a homeomorphism $D^n \cong I^n$ that takes the boundary to the boundary.)

2. In the commutative diagram below, assume that the rows are exact, that f_2 and f_4 are surjective, and that f_5 is injective.

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{g_1} & A_2 & \xrightarrow{g_2} & A_3 & \xrightarrow{g_3} & A_4 & \xrightarrow{g_4} & A_5 \\
{\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\big\downarrow \\
B_1 & \xrightarrow[h_1]{} & B_2 & \xrightarrow[h_2]{} & B_3 & \xrightarrow[h_3]{} & B_4 & \xrightarrow[h_4]{} & B_5
\end{array}$$

Show that f_3 is surjective.

3. Let $h_*(-,-)$ be a homology theory. For any excisive triad $(X; A, B)$, we have the map of long exact sequences

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & \mathrm{H}_n(A \cap B) & \xrightarrow{j_B} & \mathrm{H}_n(B) & \xrightarrow{q_B} & \mathrm{H}_n(B, A \cap B) & \xrightarrow{\delta} & \mathrm{H}_{n-1}(A \cap B) & \longrightarrow \\
& {\scriptstyle j_A}\downarrow & & \downarrow{\scriptstyle i_B} & & \downarrow\cong & & \downarrow & \\
\ldots \longrightarrow & \mathrm{H}_n(A) & \xrightarrow[i_A]{} & \mathrm{H}_n(X) & \xrightarrow[q_X]{} & \mathrm{H}_n(X, A) & \xrightarrow{\delta} & \mathrm{H}_{n-1}(A) & \longrightarrow
\end{array}$$

Use (only) this to build the Mayer-Vietoris sequence

$$\ldots \longrightarrow \mathrm{H}_n(A \cap B) \xrightarrow{(j_A, j_B)} \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \xrightarrow{i_A - i_B} \mathrm{H}_n(X) \xrightarrow{\delta} \mathrm{H}_{n-1}(A \cap B) \longrightarrow \ldots$$