Math 551 - Topology I
Homework 8
Fall 2017

1. Show that $\mathbb{R}_{\text{cocountable}}$ is connected and locally connected but not path-connected or locally path-connected. (Hint: For determining path-connectedness, it may help to first determine what are the *compact* subsets of $\mathbb{R}_{\text{coco}}$)

2. Show that $\mathbb{R}_{\text{cofinite}}$ is compact.

3. (a) Show that if X is compact and Z is any space, then the projection $p_Z : X \times Z \longrightarrow Z$ is closed.

   (b) (**Optional**) Show that the converse to (a) also holds. That is, if p_Z is closed for every Z, then X is compact.

4. (Cantor set) Let $A_0 = I = [0,1]$. Define $A_1 = A_0 \setminus (\frac{1}{3}, \frac{2}{3})$. Similarly, define A_2 by removing the middle thirds of the intervals in A_1:

$$A_2 = A_1 \setminus \left(\left(\frac{1}{9}, \frac{2}{9} \right) \cup \left(\frac{7}{9}, \frac{8}{9} \right) \right).$$

   In general, given A_n constructed in this way, we define A_{n+1} by removing the middle thirds of all intervals in A_n. Define the Cantor set to be

$$C = \bigcap_n A_n \subseteq [0,1].$$

   (a) Show that C is compact (without using part (d)).

   (b) Show that C is totally disconnected (every connected component is a singleton).

   (c) Show that any compact, locally connected space has finitely many components. Conclude that C is not locally connected.

   (d) Let $D = \{0, 2\}$ with the discrete topology. Show that $C \cong \prod_n D$. (Hint: instead of binary expansions, think about ternary expansions of numbers in $[0,1]$.)

5. Let X be Hausdorff, and suppose that $C, D \subseteq X$ are disjoint compact subsets. Show that there are disjoint open sets $U, V \subseteq X$ with $C \subseteq U$ and $D \subseteq V$.