Mon, Oct. 23

16.1. **Path Connectedness.** Ok, so we have looked at examples and studied this notion of being connected, but if you asked your calculus students to describe what it should mean for a subset of $\mathbb{R}$ to be connected, they probably wouldn't come up with the "no nontrivial clopen subsets" idea. Instead, they would probably say something about being able to connect-the-dots. In other words, you should be able to draw a line from one point to another while staying in the subset. This leads to the following idea.

Definition 16.12. We say that $A \subseteq X$ is **path-connected** if for every pair a, b of points in A, there is a continuous function (a path) $\gamma : I \longrightarrow A$ with $\gamma(0) = a$ and $\gamma(1) = b$.

This is not unrelated to the earlier notion.

Proposition 16.13. *If $A \subseteq X$ is path-connected, then it is also connected.*

Proof. Pick a point $a_0 \in A$. For any other $b \in A$, we have a path γ_b in A from a_0 to b. Then the image $\gamma_b(I)$ is a connected subset of A containing both a_0 and b. It follows that

$$A = \bigcup_{b \in A} \gamma_b(I)$$

is connected, as it is the overlapping union of connected sets. ■

For subsets $A \subseteq \mathbb{R}$, we have

$$A \text{ is path-connected} \Rightarrow A \text{ is connected} \Leftrightarrow A \text{ is an interval} \Rightarrow A \text{ is path-connected}.$$

So the two notions coincide for subsets of $\mathbb{R}$. But the same is not true in $\mathbb{R}^2$! (The topologist's sine curve, HW 7).

Path-connectedness has much the same behavior as connectedness.

Proposition 16.14.

(1) *Images of path-connected spaces are path-connected*
(2) *Overlapping unions of path-connected spaces are path-connected*
(3) *Finite products of path-connected spaces are path-connected*

However, the topologist's sine curve shows that closures of path-connected subsets need not be path-connected.

Our proof of connectivity of $\prod_i X_i$ last time used this closure property for connected sets, so the earlier argument does not adapt easily to path-connectedness. But it turns out to be easier to prove.

Theorem 16.15. *Assume $X_i \neq \emptyset$ for all $i \in I$, where is I is arbitrary. Then $\prod_i X_i$ is path-connected if and only if each X_i is path-connected.*

Proof. The interesting direction is ($\Leftarrow$). Thus assume that each X_i is path-connected. Let (x_i) and (y_i) be points in the product $\prod_i X_i$. Then for each $i \in \mathcal{I}$ there is a path γ_i in X_i with $\gamma_i(0) = x_i$ and $\gamma_i(1) = y_i$. By the universal property of the product, we get a continuous path

$$\gamma = (\gamma_i) : [0, 1] \longrightarrow \prod_i X_i$$

with $\gamma(0) = (x_i)$ and $\gamma(1) = (y_i)$. ■

16.2. **Components.** The overlapping union property for (path-)connectedness allows us to make the following definition.

Definition 16.16. Let $x \in X$. We define the **connected component** (or simply component) of x in X to be

$$C_x = \bigcup_{\substack{x \in C \\ \text{connected}}} C.$$

Similarly, the **path-component** of X is defined to be

$$PC_x = \bigcup_{\substack{x \in P \\ \text{connected}}} P.$$

The overlapping union property guarantees that C_x is connected and that PC_x is path-connected. Since path-connected sets are connected, it follows that for any x, we have $PC_x \subseteq C_x$. An immediate consequence of the above definition(s) is that any (path-)connected subset of X is contained in some (path-)component.

Example 16.17. Consider $\mathbb{Q}$, equipped with the subspace topology from $\mathbb{R}$. Then the only connected subsets are the singletons, so $C_x = \{x\}$. Such a space is said to be **totally disconnected**.

Note that for any space X, each component C_x is closed as $\overline{C_x}$ is a connected subset containing x, which implies $\overline{C_x} \subseteq C_x$. If X has finitely many components, then each component is the complement of the finite union of the remaining components, so each component is also open, and X decomposes as a disjoint union

$$X \cong C_1 \amalg C_2 \amalg \cdots \amalg C_n$$

of its components. But this does not happen in general, as the previous example shows.

The situation is worse for path-components: they need not be open or closed, as the topologist's sine curve shows.

16.3. **Locally (Path-)Connected.**

Definition 16.18. Let X be a space. We say that X is **locally connected** if any neighborhood U of any point x contains a connected neighborhood $x \in V \subset U$. Similarly X is **locally path-connected** if any neighborhood U of any point x contains a path-connected neighborhood $x \in V \subset U$.

The locally path-connected turns out to show up more often, so we focus on that.

Proposition 16.19. *Let X be a space. The following are equivalent.*

1. *X is locally path-connected*
2. *X has a basis consisting of path-connected open sets*
3. *for every open set $U \subseteq X$, the path-components of U are open in X*
4. *for every open set $U \subseteq X$, every component of U is path-connected and open in X.*

Proof. We leave the implications (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) as an exercise. We argue for (1) $\Leftrightarrow$ (4).

Assume X is locally path-connected, and let C be a component of an open subset $U \subseteq X$. Let $P \subseteq C$ be a nonempty path-component. Then P is open in X. But all of the other path-components of C are also open, so their union, which is the complement of P, must be open. It follows that P is closed. Since C is connected, we must have $P = C$.

On the other hand, suppose that (4) holds. Let U be a neighborhood of x. Then the component C_x of x in U is the desired neighborhood V. ■

In particular, this says that the components and path-components agree if X is locally path-connected.

Just as path-connected implies connected, locally path-connected implies locally-connected. But, unfortunately, there are no other implications between the four properties.

Example 16.20. The topologist's sine curve is connected, but not path-connected or locally connected or locally path-connected. Thus it is possible to be connected but not locally so.

Example 16.21. For any space X, the **cone** on X is defined to be $CX = X \times [0,1]/X \times \{1\}$. The cone on any space is always path-connected. In particular, the cone on the topologist's sine curve is connected and-path connected but not locally connected or locally path-connected.

Example 16.22. A disjoint union of two topologist's sine curves gives an example that is not connected in any of the four ways.

Example 16.23. Note that if X is locally path-connected, then connectedness is equivalent to path-connectedness. A connected example would be $\mathbb{R}$ or a one-point space. A disconnected example would be $(0,1) \cup (2,3)$ or a two point (discrete) space.

Finally, we have spaces that are locally connected but not locally path-connected.

Example 16.24. The cocountable topology on $\mathbb{R}$ is connected and locally connected but not path-connected or locally path-connected.

Example 16.25. The cone on the cocountable topology will give a connected, path-connected, locally connected space that is not locally path-connected.

Example 16.26. Two copies of $\mathbb{R}_{\text{cocountable}}$ give a space that is locally connected but not connected in the other three ways.

Wed, Oct. 25

17. Compactness

The next topic is one of the major ones in the course: compactness. As we will see, this is the analogue of a "closed and bounded subset" in a general space. The definition relies on the idea of coverings.

Definition 17.1. An **open cover** of X is a collection $\mathcal{U}$ of open subsets that cover X. In other words, $\bigcup_{U \in \mathcal{U}} U = X$. Given two covers $\mathcal{U}$ and $\mathcal{V}$ of X, we say that $\mathcal{V}$ is a **subcover** if $\mathcal{V} \subseteq \mathcal{U}$.

Definition 17.2. A space X is said to be **compact** if every open cover has a *finite* subcover (i.e. a cover involving finitely many open sets).

Example 17.3. Clearly any finite topological space is compact, no matter the topology.

Example 17.4. An infinite set with the discrete topology is *not* compact, as the collection of singletons gives an open cover with no finite subcover.

Example 17.5. $\mathbb{R}$ is not compact, as the open cover $\mathcal{U} = \{(-n,n) \mid n \in \mathbb{N}\}$ has no finite subcover.

Example 17.6. Similarly $[0,\infty)$ is not compact, as the open cover $\mathcal{U} = \{[0,n)\}$ has no finite subcover. Recall that $[0,\infty) \cong [a,b)$.

Theorem 17.7. *Let $a < b$. Then $[a,b]$ is a compact subset of $\mathbb{R}$.*

Proof. Let $\mathcal{U}$ be an open cover. Then some element of the cover must contain a. Pick such an element and call it U_0.

Consider the set

$$\mathcal{E} = \{c \in [a,b] \mid [a,c] \text{ is finitely covered by } \mathcal{U}\}.$$

Certainly $a \in \mathcal{E}$ and $\mathcal{E}$ is bounded above by b. By the Least Upper Bound Axiom, $s = \sup \mathcal{E}$ exists. Note that $a \leq s \leq b$, so we must have $s \in U_s$ for some $U_s \in \mathcal{U}$. In general, U_s may not be connected, so let $s \in V \subseteq U_s$ be an open interval. But then for any $c < s$ with $c \in V \subseteq U_s$, we have $c \in \mathcal{E}$. This means that

$$[a,c] \subseteq U_1 \cup \cdots \cup U_k$$

for $U_1, \ldots, U_k \in \mathcal{U}$. But then $[a,s] \subseteq U_1 \cup \cdots \cup U_k \cup U_s$. This shows that $s \in \mathcal{E}$. On the other hand, the same argument shows that for any $s < d < b$ with $d \in U_s$, we would similarly have $d \in \mathcal{E}$. Since $s = \sup \mathcal{E}$, there cannot exist such a d. This implies that $s = b$. ■

Like connectedness, compactness is preserved by continuous functions.

Proposition 17.8. *Let $f : X \longrightarrow Y$ be continuous, and assume that X is compact. Then $f(X)$ is compact.*

Proof. Let $\mathcal{V}$ be an open cover of $f(X)$. Then $\mathcal{U} = \{f^{-1}(V) \mid V \in \mathcal{V}\}$ is an open cover of X. Let $\{U_1, \ldots, U_k\}$ be a finite subcover. It follows that the corresponding $\{V_1, \ldots, V_k\}$ is a finite subcover of $\mathcal{V}$. ■

Fri, Oct 27

Example 17.9. Recall that we have the quotient map $\exp : [0,1] \longrightarrow S^1$. It follows that S^1 is compact.

Theorem 17.10 (Extreme Value Theorem)**.** *Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous. Then f attains a maximum and a minimum.*

Proof. Since f is continuous and $[a,b]$ is both connected and compact, the same must be true of its image. But the compact, connected subsets are precisely the closed intervals. ■

The following result is also quite useful.

Proposition 17.11. *Let X be Hausdorff and let $A \subseteq X$ be a compact subset. Then A is closed in X.*

Proof. Pick any point $x \in X \setminus A$ (if we can't, then $A = X$ and we are done). For each $a \in A$, we have disjoint neighborhoods $a \in U_a$ and $x \in V_a$. Since the U_a cover A, we only need finitely many, say $U_{a_1}, \ldots, U_{a_k}$ to cover A. But then the intersection

$$V = V_{a_1} \cap \cdots \cap V_{a_k}$$

of the corresponding V_a's is disjoint from the union of the U_a's and therefore also from A. Since V is a finite intersection of open sets, it is open and thus gives a neighborhood of x in $X \setminus A$. It follows that A is closed. ■

Exercise 17.12. *If $A \subseteq X$ is closed and X is compact, then A is compact.*

Combining these results gives the following long-awaited consequence.

Corollary 17.13. *Let $f : X \longrightarrow Y$ be continuous, where X is compact and Y is Hausdorff, then f is a closed map.*

In particular, if f is already known to be a continuous bijection, then it is automatically a homeomorphism. For example, this shows that the map $I/\partial I \longrightarrow S^1$ is a homeomorphism. Similarly, from Example 14.10 we have $D^n/\partial D^n \cong S^n$.

17.1. **Products.** We will next show that finite products of compact spaces are compact, but we first need a lemma.

Lemma 17.14 (Tube Lemma). *Let X be compact and Y be any space. If $W \subseteq X \times Y$ is open and contains $X \times \{y\}$, then there is a neighborhood V of y with $X \times V \subseteq W$.*

Proof. For each $x \in X$, we can find a basic neighborhood $U_x \times V_x$ of (x, y) in W. The U_x's give an open cover of X, so we only need finitely many of them, say $U_{x_1}, \ldots, U_{x_n}$. Then we may take $V = V_{x_1} \cap \cdots \cap V_{x_n}$. ■