Mon, Oct. 30

Proposition 17.15. *Let X and Y be nonempty. Then $X \times Y$ is compact if and only if X and Y are compact.*

Proof. As for connectedness, the continuous projections make X and Y compact if $X \times Y$ is compact.

Now suppose that X and Y are compact and let $\mathcal{U}$ be an open cover. For each $y \in Y$, the cover $\mathcal{U}$ of $X \times Y$ certainly covers the slice $X \times \{y\}$. This slice is homeomorphic to X and therefore finitely-covered by some $\mathcal{V} \subset \mathcal{U}$. By the Tube Lemma, there is a neighborhood V_y of y such that the tube $X \times V_y$ is covered by the same $\mathcal{V}$. Now the V_y's cover Y, so we only need finitely many of these to cover X. Since each tube is finitely covered by $\mathcal{U}$ and we can cover $X \times Y$ by finitely many tubes, it follows that $\mathcal{U}$ has a finite subcover. ■

17.2. **Compactness in $\mathbb{R}^n$.**

Theorem 17.16 (Heine-Borel)**.** *A subset $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded (contained in a single metric ball).*

Proof. Suppose A is compact. Then A must be closed in $\mathbb{R}^n$ since $\mathbb{R}^n$ is Hausdorff. To see that A is bounded, pick any point $a \in A$ (if A is empty, we are certainly done). Then the collection of balls $B_n(a) \cap A$ gives an open cover of A, since any other point in A is a finite distance away from a. Since A is compact, there must be a finite subcover $\{B_{n_1}(a), \ldots, B_{n_k}(a)\}$. Let $N = \max\{n_1, \ldots, n_k\}$. Then $A \subseteq B_N(a)$.

On the other hand, suppose that A is closed and bounded in $\mathbb{R}^n$. Since A is bounded, it is contained in $[-k, k]^n$ for some $k > 0$. But this product of intervals is compact since each interval is compact. Now A is a closed subset of a compact space, so it is compact. ■

In fact, the forward implication of the above proof works to show that

Proposition 17.17. *Let $A \subseteq X$, where X is metric and A is compact. Then A is closed and bounded in X.*

But the reverse implication is not true in general, as the next example shows.

Example 17.18. Consider $[0, \pi] \cap \mathbb{Q} \subseteq \mathbb{Q}$. This is certainly closed and bounded, but we will see it is not compact. Consider the open cover $\mathcal{U} = \left\{[0, \pi - \frac{1}{n}) \cap \mathbb{Q}\right\}_{n \in \mathbb{N}}$. This has no finite subcover.

Again, we have shown that compactness interacts well with finite products, and we would like a similar result in the arbitrary product case. This is a major theorem, known as the Tychonoff theorem. First, we show the theorem does not hold with the box topology.

Example 17.19. Consider $X = \{0, 1\}^{\mathbb{N}}$. In the box topology, this is discrete. Since this is infinite, it cannot be compact.

Example 17.20. We have studied the orthogonal subgroups $O(n) \subseteq Gl_n(\mathbb{R})$. The bigger group $Gl_n(\mathbb{R})$ is not compact, as it is neither closed nor bounded in $\mathbb{R}^{n^2}$. However, the orthogonality relations defining orthogonal matrices make this a closed subset of $\mathbb{R}^{n^2}$, and the fact that each column has norm 1 means that an orthogonal $n \times n$ matrix, when considered as a point in $\mathbb{R}^{n^2}$, has norm $\sqrt{n}$. In particular, $O(n)$ is a bounded subset of $\mathbb{R}^{n^2}$.

17.3. **Tychonoff's Theorem.**

Theorem 17.21 (Tychonoff)**.** *Let $X_i \neq \emptyset$ for all $i \in \mathcal{I}$. Then $\prod_i X_i$ is compact if and only if each X_i is compact.*

Our proof, even for the difficult direction, will use the axiom of choice. In fact, Tychonoff's theorem is equivalent to the axiom of choice.

Theorem 17.22. *Tychonoff $\Rightarrow$ axiom of choice.*

Proof. This argument is quite a bit simplier than the other implication. Let $X_i \neq \emptyset$ for all $i \in \mathcal{I}$. We want to show that $X = \prod_i X_i \neq \emptyset$.

For each i, define $Y_i = X_i \cup \{\infty_i\}$, where $\infty_i \notin X_i$. We topologize Y_i such that the only nontrivial open sets are X_i and $\{\infty_i\}$. Now for each i, let $U_i = p_i^{-1}(\infty_i)$. The collection $\mathcal{U} = \{U_i\}$ gives a collection of open subsets of $Y = \prod_i Y_i$, and this collection covers Y if and only if $X = \emptyset$. Each Y_i is compact since it has only four open sets. Thus Y must be compact by the Tychonoff theorem. But no finite subcollection of $\mathcal{U}$ can cover Y. For example, $U_i \cup U_j$ does not cover Y since if $a \in X_i$ and $b \in X_j$, then we can define $(y_i) \in Y \setminus (U_i \cup U_j)$ by

$$y_k = \begin{cases} a & k = i \\ b & k = j \\ \infty_k & k \neq i, j \end{cases}$$

The same kind of argument will work for any finite collection of U_i's. Since $\mathcal{U}$ has no finite subcover and Y is compact, $\mathcal{U}$ cannot cover Y, so that X must be nonempty. ■

Wed, Nov. 1

The Tychonoff Theorem is *equivalent* to the axiom of choice. We will thus use a form of the axiom of choice in order to prove it.

Zorn's Lemma. Let P be a partially-ordered set. If every linearly-ordered subset of P has an upper bound in P, then P contains at least one maximal element.

Theorem 17.23 (Tychonoff)**.** *Let $X_i \neq \emptyset$ for all $i \in \mathcal{I}$. Then $\prod_i X_i$ is compact if and only if each X_i is compact.*

Proof. As we have seen a number of times, the implication ($\Rightarrow$) is trivial.

We now show the contrapositive of ($\Leftarrow$). Thus assume that $X = \prod_i X_i$ is not compact. We wish to conclude that one of the X_i must be noncompact. By hypothesis, there exists an open cover $\mathcal{U}$ of X with no finite subcover.

We first deal with the following case.

Special case: *$\mathcal{U}$ is a cover by prebasis elements.*
For each $i \in \mathcal{I}$, let $\mathcal{U}_i$ be the collection

$$\mathcal{U}_i = \{V \subseteq X_i \text{ open } \mid p_i^{-1}(V) \in \mathcal{U}\}.$$

For some i, the collection $\mathcal{U}_i$ must cover X_i, since otherwise we could pick $x_i \in X_i$ for each i with x_i not in the union of $\mathcal{U}_i$. Then the element $(x_i) \in \prod_i X_i$ would not be in $\mathcal{U}$ since it cannot be in any $p_i^{-1}(V)$. But now the cover $\mathcal{U}_i$ cannot have a finite subcover, since that would provide a corresponding subcover of $\mathcal{U}$. It follows that X_i is not compact.

It remains to show that we can always reduce to the cover-by-prebasis case.

Consider the collection $\mathcal{N}$ of open covers of X having no finite subcovers. By assumption, this set is nonempty, and we can partially order $\mathcal{N}$ by inclusion of covers. Furthermore, if $\{\mathcal{U}_\alpha\}$ is a linearly order subset of $\mathcal{N}$, then $\mathcal{U} = \bigcup_\alpha \mathcal{U}_\alpha$ is an open cover, and it cannot have a finite subcover since a finite subcover of $\mathcal{U}$ would be a finite subcover of one of the $\mathcal{U}_\alpha$. Thus $\mathcal{U}$ is an upper bound in $\mathcal{N}$ for $\{\mathcal{U}_\alpha\}$. By Zorn's Lemma, $\mathcal{N}$ has a maximal element $\mathcal{V}$.

Now let $\mathcal{S} \subseteq \mathcal{V}$ be the subcollection consisting of the prebasis elements in $\mathcal{V}$. We claim that $\mathcal{S}$ covers X. *Suppose not.* Thus let $x \in X$ such that x is not covered by $\mathcal{S}$. Then x must be in V for some $V \in \mathcal{V}$. By the definition of the product topology, x must have a basic open neighborhood in $B \subset V$. But any basic open set is a finite intersection of prebasic open sets, so $B = S_1 \cap \dots S_k$. If x is not covered by $\mathcal{S}$, then none of the S_i are in $\mathcal{S}$. Thus $\mathcal{V} \cup \{S_i\}$ is not in $\mathcal{N}$ by maximality of $\mathcal{V}$. In other words, $\mathcal{V} \cup \{S_i\}$ has a finite subcover $\{V_{i,1}, \dots, V_{i,n_i}, S_i\}$. Let us write

$$\hat{V}_i = V_{i,1} \cup \dots \cup V_{i,n_i}.$$

Now

$$X = \bigcap_i \left(S_i \cup \hat{V}_i\right) \subseteq \left(\bigcap_i S_i\right) \cup \left(\bigcup_i \hat{V}_i\right) \subseteq V \cup \left(\bigcup_i \hat{V}_i\right)$$

This shows that $\mathcal{V}$ has a finite subcover, which contradicts that $\mathcal{V} \in \mathcal{N}$. We thus conclude that $\mathcal{S}$ covers X using only prebasis elements.

But now by the argument at the beginning of the proof, $\mathcal{S}$, and therefore $\mathcal{V}$ as well, has a finite subcover. This is a contradiction. ∎

$\boxed{\textbf{Fri, Nov. 3}}$

Remark 17.24. There are other versions of compactness. For instance **sequential compactness** is the condition that every sequence has a convergent subsequence. In a metric space, this turns out to be equivalent to compactness, but not for general topological spaces.

17.4. Local Compactness.

Definition 17.25. We say that a space is **locally compact** if every $x \in X$ has a compact neighborhood (recall that we do not require neighborhoods to be open).

This looks different from our other "local" notions. To get a statement in the form we expect, we introduce more terminology $A \subseteq X$ is **precompact** if $\overline{A}$ is compact.

Proposition 17.26. *Let X be Hausdorff. TFAE*

1. *X is locally compact*
2. *every $x \in X$ has a precompact neighborhood*
3. *X has a basis of precompact open sets*

Proof. It is clear that (3) ⇒ (2) ⇒ (1) without the the Hausdorff assumption, so we show that (1) ⇒ (3). Suppose X is locally compact and Hausdorff. Let V be open in X and let $x \in V$. We want a precompact open neighborhood of x in V. Since X is locally compact, we have a compact neighborhood K of x, and since X is Hausdorff, K must be closed. Since V and K are both neighborhoods of x, so is $V \cap K$. Thus let $x \in U \subseteq V \cap K$. Then $\overline{U} \subseteq K$ since K is closed, and $\overline{U}$ is compact since it is a closed subset of a compact set. ∎

In contrast to the local connectivity properties, it is clear that any compact space is locally compact. But this is certainly a generalization of compactness, since *any interval* in $\mathbb{R}$ is locally compact.

Example 17.27. A standard example of a space that is not locally compact is $\mathbb{Q} \subseteq \mathbb{R}$. We show that 0 does not have any compact neighborhoods . Let V be any neighborhood of 0. Then it must contain $(-\pi/n, \pi/n)$ for some n. Now

$$\mathcal{U} = \left\{ \left(-\pi/n, \left(\frac{k}{k+1}\right)\pi/n \right) \right\} \cup \left\{ V \cap (\pi/n, \infty), V \cap (-\infty, -\pi/n) \right\}$$

is an open cover of V with no finite subcover.

Remark 17.28. Why did we define local compactness in a different way from local (path)-connectedness? We could have defined locally connected to mean that every point has a connected neighborhood , which follows from the actual definition. But then we would not have that locally connected is equivalent to having a basis of connected open sets. On the other hand, we could try the $x \in K \subseteq U$ version of locally compact, but of course we don't want to allow $K = \{x\}$, so the next thing to require is $x \in V \subseteq U$, where V is precompact. As we showed in Prop 17.26, this is equivalent to our definition of locally compact in the presence of the Hausdorff condition. Without the Hausdorff condition, compactness does not behave quite how we expect.

18. Compactification

Locally compact Hausdorff spaces are a very nice class of spaces (almost as good as compact Hausdorff). In fact, any such space is close to a compact Hausdorff space.

Definition 18.1. A **compactification** of a noncompact space X is an embedding $i : X \hookrightarrow Y$, where Y is compact and $i(X)$ is dense.

We will typically work with Hausdorff spaces X, in which case we ask the compactification Y to also be Hausdorff.

Example 18.2. The open interval $(0, 1)$ is not compact, but $(0, 1) \hookrightarrow [0, 1]$ is a compactification. Note that the exponential map $\exp : (0, 1) \longrightarrow S^1$ also gives a (different) compactification. The topologist's sine curve (HW 7.5) also gives a (much larger) compactification.