$\boxed{\textbf{Mon, Nov. 6}}$

There is often a smallest compactification, given by the following construction.

Definition 18.3. Let X be a space and define $\widehat{X} = X \cup \{\infty\}$, where $U \subseteq \widehat{X}$ is open if either

- $U \subseteq X$ and U is open in X or
- $\infty \in U$ and $\widehat{X} \setminus U \subseteq X$ is compact.

Proposition 18.4. *Suppose that X is Hausdorff and noncompact. Then $\widehat{X}$ is a compactification. If X is locally compact, then $\widehat{X}$ is Hausdorff.*

Proof. We first show that $\widehat{X}$ is a space! It is clear that both $\emptyset$ and $\widehat{X}$ are open.

Suppose that U_1 and U_2 are open. We wish to show that $U_1 \cap U_2$ is open.

- If neither open set contains ∞, this follows since X is a space.
- If $\infty \in U_1$ but $\infty \notin U_2$, then $K_1 = X \setminus U_1$ is compact. Since X is Hausdorff, K_1 is closed in X. Thus $X \setminus K_1 = U_1 \setminus \{\infty\}$ is open in X, and it follows that $U_1 \cap U_2 = (U_1 \setminus \{\infty\}) \cap U_2$ is open.
- If $\infty \in U_1 \cap U_2$, then $K_1 = X \setminus U_1$ and $K_2 = X \setminus U_2$ are compact. It follows that $K_1 \cup K_2$ is compact, so that $U_1 \cap U_2 = X \setminus (K_1 \cup K_2)$ is open.
- Suppose we have a collection U_i of open sets. If none contain ∞, then neither does $\bigcup\limits_i U_i$, and the union is open in X. If $\infty \in U_j$ for some j, then $\infty \in \bigcup\limits_i U_i$ and
$$\widehat{X} \setminus \bigcup_i U_i = \bigcap_i (\widehat{X} \setminus U_i) = \bigcap_i (X \setminus U_i)$$
is a closed subset of the compact set $X \setminus U_j$, so it must be compact.

Next, we show that $\iota : X \longrightarrow \widehat{X}$ is an embedding. Continuity of ι again uses that compact subsets of X are closed. That ι is open follows immediately from the definition of $\widehat{X}$.

To see that $\iota(X)$ is dense in $\widehat{X}$, it suffices to see that $\{\infty\}$ is not open. But this follows from the definition of $\widehat{X}$, since X is not compact.

Finally, we show that $\widehat{X}$ is compact. Let $\mathcal{U}$ be an open cover. Then some $U \in \mathcal{U}$ must contain ∞. The remaining elements of $\mathcal{U}$ must cover $X \setminus U$, which is compact. It follows that we can cover $X \setminus U$ using only finitely many elements, so $\mathcal{U}$ has a finite subcover.

Now suppose that X is locally compact. Let x_1 and x_2 in $\widehat{X}$. If neither is ∞, then we have disjoint neighborhoods in X, and these are still disjoint neighborhoods in $\widehat{X}$. If $x_2 = \infty$, let $x_1 \in U \subseteq K$, where U is open and K is compact. Then U and $V = \widehat{X} \setminus K$ are the desired disjoint neighborhoods. ■

Example 18.5. We saw that S^1 is a one-point compactification of $(0, 1) \cong \mathbb{R}$. You will show on your homework that similarly S^n is a one-point compactification of $\mathbb{R}^n$.

Example 18.6. As we have seen, $\mathbb{Q}$ is not locally compact, so we do not expect $\widehat{\mathbb{Q}}$ to be Hausdorff. Indeed, any open subset containing ∞ is dense in $\widehat{\mathbb{Q}}$. Because of the topology on $\widehat{\mathbb{Q}}$, this is equivalent to showing that for any open, nonempty subset $U \subseteq \mathbb{Q}$, U is not contained in any compact subset. Since $\mathbb{Q}$ is Hausdorff, if U were contained in a compact subset, then $\overline{U}$ would also be compact. But as we have seen, for any interval $(a, b) \cap \mathbb{Q}$, the closure in $\mathbb{Q}$, which is $[a, b] \cap \mathbb{Q}$, is not compact.

Next, we show that the situation we observed for compactifications of $(0, 1)$ holds quite generally.

$$\begin{tikzcd} Y \arrow[rr, dashed, "q"] & & \widehat{X} \\ & X \arrow[ul, "f"] \arrow[ur, "\iota"'] & \end{tikzcd}$$

Proposition 18.7. *Let X be locally compact Hausdorff and let $f : X \longrightarrow Y$ be a (Hausdorff) compactification. Then there is a (unique) quotient map $q : Y \longrightarrow \widehat{X}$ such that $q \circ f = \iota$.*

We will need:

Lemma 18.8. *Let X be locally compact Hausdorff and $f : X \longrightarrow Y$ a compactification. Then f is open.*

Proof of Prop. 18.7. We define

$$q(y) = \begin{cases} \iota(x) & \text{if } y = f(x) \\ \infty & \text{if } y \notin f(X). \end{cases}$$

To see that q is continuous, let $U \subseteq \widehat{X}$ be open. If $\infty \notin U$, then $q^{-1}(U) = f(\iota^{-1}(U))$ is open by the lemma. If $\infty \in U$, then $K = \widehat{X} \setminus U$ is compact and thus closed. We have $q^{-1}(K) = f(\iota^{-1}(K))$ is compact and closed in Y, so it follows that $q^{-1}(U) = Y \setminus q^{-1}(K)$ is open.

Note that q is automatically a quotient map since it is a closed continuous surjection (it is closed because Y is compact and $\widehat{X}$ is Hausdorff). Note also that q is unique because $\widehat{X}$ is Hausdorff and q is already specified on the dense subset $f(X) \subseteq Y$. ■

Wed, Nov. 8

Last time, we said that we had a *unique* quotient map $q : Y \longrightarrow \widehat{X}$ for any Hausdorff compactification Y. Why is it unique? The definition of q on the dense subset $f(X) \subset Y$ was forced, and $\widehat{X}$ is Hausdorff. Then uniqueness is given by

Proposition 18.9. *Let Z be Hausdorff, and let $f, g : X \rightrightarrows Z$ be continuous functions. If f and g agree on a dense subset, then they agree on all of X.*

Proof of Lemma. Since f is an emebedding, we can pretend that $X \subseteq Y$ and that f is simply the inclusion. We wish to show that X is open in Y. Thus let $x \in X$. Let U be a precompact neighborhood of x. Thus $K = \mathrm{cl}_X(U)$ is compact[3] and so must be closed in Y (and X) since Y is Hausdorff. By the definition of the subspace topology, we must have $U = V \cap X$ for some open $V \subseteq Y$. Then V is a neighborhood of x in Y, and

$$V = V \cap Y = V \cap \mathrm{cl}_Y(X) \subseteq \mathrm{cl}_Y(V \cap X) = K \subseteq X.$$

The middle inclusion can be checked using the neighborhood criterion, using that V is open in Y. ■

Corollary 18.10. *Any two one-point compactifications are homeomorphic.*

The following is a useful characterization of locally compact Hausdorff spaces.

Proposition 18.11. *A space X is Hausdorff and locally compact if and only if it is homeomorphic to an open subset of a compact Hausdorff space Y.*

Proof. ($\Rightarrow$). We saw that X is open in the compact Hausdorff space $Y = \widehat{X}$.

($\Leftarrow$) As a subspace of a Hausdorff space, X must also be Hausdorff. It remains to show that every point has a compact neighborhood (in X). Write $Y_\infty = Y \setminus X$. This is closed in Y and therefore compact. By Problem 3 from HW7, we can find disjoint open sets $x \in U$ and $Y_\infty \subseteq V$ in Y. Then $K = Y \setminus V$ is the desired compact neighborhood of x in X. ■

[3]We will need to distinguish between closures in X and closures in Y, so we use the notation $\mathrm{cl}_X(A)$ for closure rather than our usual $\overline{A}$.

Corollary 18.12. *If X and Y are locally compact Hausdorff, then so is $X \times Y$.*

Corollary 18.13. *Any open or closed subset of a locally compact Hausdorff space is locally compact Hausdorff.*

18.1. **Separation Axioms.** We finally turn to the so-called "separation axioms".

Definition 18.14. A space X is said to be

- T_0 if given two distinct points x and y, there is a neighborhood of one not containing the other
- T_1 if given two distinct points x and y, there is a neighborhood of x not containing y and vice versa (points are closed)
- T_2 (**Hausdorff**) if any two distinct points x and y have disjoint neighborhoods
- T_3 (**regular**) if *points are closed and* given a closed subset A and $x \notin A$, there are disjoint open sets U and V with $A \subseteq U$ and $x \in V$
- T_4 (**normal**) if *points are closed and* given closed disjoint subsets A and B, there are disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.

Note that $T_4 \implies T_3 \implies T_2 \implies T_1 \implies T_0$. But beware that in some literature, the "points are closed" clause is not included in the definition of regular or normal. Without that, we would not be able to deduce T_2 from T_3 or T_4.

We have talked a lot about Hausdorff spaces. The other important separation property is T_4. We will not really discuss the intermediate notion of regular (or the other variants completely regular, completely normal, etc.)

Proposition 18.15. *Any compact Hausdorff space is normal.*

Proof. This was homework problem 8.5. ■

More generally,

Theorem 18.16. *Suppose X is locally compact, Hausdorff, and second-countable. Then X is normal.*

Another important class of normal spaces is the collection of metric spaces.

Proposition 18.17. *If X is metric, then it is normal.*

Unfortunately, the T_4 condition alone is not preserved by the constructions we have studied.

Example 18.18. (Images) $\mathbb{R}$ is normal. But recall the quotient map $q : \mathbb{R} \longrightarrow \{-1, 0, 1\}$ which sends any number to its sign. This quotient is not Hausdorff and therefore not (regular or) normal.

Example 18.19. (Subspaces) If J is uncountable, then the product $(0,1)^J$ is *not* normal (Munkres, example 32.2). This is a subspace of $[0,1]^J$, which is compact Hausdorff by the Tychonoff theorem and therefore normal. So a subspace of a normal space need not be normal. We also saw in this example that (uncountable) products of normal spaces need not be normal.

Ok, so we've seen a few examples. So what, why should we care about normal spaces? Look back at the definition for T_2, T_3, T_4. In each case, we need to find certain open sets U and V. How would one do this in general? In a metric space, we would build these up by taking unions of balls. In an arbitrary space, we might use a basis. But another way of getting open sets is by pulling back open sets under a continuous map. That is, suppose we have a map $f : X \longrightarrow [0,1]$ such that $f \equiv 0$ on A and $f \equiv 1$ on B. Then $A \subseteq U := f^{-1}([0, \frac{1}{2}))$ and $B \subseteq V := f^{-1}((\frac{1}{2}, 1])$, and $U \cap V = \emptyset$.

One of the main consequences of normality is

Theorem 18.20 (Urysohn's Lemma)**.** *Let X be normal and let A and B be disjoint closed subsets. Then there exists a continuous function $f : X \longrightarrow [0,1]$ such that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$.*

Note that Urysohn's Lemma becomes an if and only if statement if we either drop the T_1-condition from normal or if we explicitly include singletons as possible replacements for A and B.

A typical application of Urysohn's lemma is to create **bump functions**, which are equal to 1 on a closed set A and vanish outside some open $U \supset A$.

Theorem 18.21. *Suppose X is locally compact, Hausdorff, and second-countable. Then X is metrizable.*

See [Munkres, Theorem 34.1]. The point is that you can use Urysohn functions to give an embedding of X into $\mathbb{R}^{\mathbb{N}}$.

Part 5. Nice spaces - the ones we really, really care about

Fri, Nov. 10

19. Manifolds

We finally arrive at one of the most important definitions of the course.

Definition 19.1. A (topological) ***n*-manifold** M is a Hausdorff, second-countable space such that each point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

Example 19.2.
(1) $\mathbb{R}^n$ and any open subset is obviously an n-manifold
(2) S^1 is a 1-manifold. More generally, S^n is an n-manifold. Indeed, we have shown that if you remove a point from S^n, the resulting space is homeomorphic to $\mathbb{R}^n$.
(3) T^n, the n-torus, is an n-manifold. In general, if M is an m-manifold and N is an n-manifold, then $M \times N$ is an $(m+n)$-manifold.
(4) $\mathbb{RP}^n$ is an n-manifold. There is a standard covering of $\mathbb{RP}^n$ by open sets as follows. Recall that $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^\times$. For each $1 \leq i \leq n+1$, let $V_i \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ be the complement of the hyperplane $x_i = 0$. This is an open, saturated set, and so its image $U_i = V_i/\mathbb{R}^\times \subseteq \mathbb{RP}^n$ is open. The V_i's cover $\mathbb{R}^{n+1} \setminus \{0\}$, so the U_i's cover $\mathbb{RP}^n$. We leave the rest of the details as an exercise.
(5) $\mathbb{CP}^n$ is a $2n$-manifold. This is similar to the description given above.
(6) $O(n)$ is a $\frac{n(n-1)}{2}$-manifold. Since it is also a topological group, this makes it a *Lie group.* The standard way to see that this is a manifold is to realize the orthogonal group as the preimage of the identity matrix under the transformation $M_n(R) \longrightarrow M_n(R)$ that sends A to $A^T A$. This map lands in the subspace $S_n(R)$ of symmetric $n \times n$ matrices. This space can be identified with $\mathbb{R}^{n(n+1)/2}$.

Now the $n \times n$ identity matrix is an element of S_n, and an important result in differential topology (Sard's theorem) that says that if a certain derivative map is surjective, then the preimage of the submanifold $\{I_n\}$ will be a submanifold of $M_n(\mathbb{R})$ of the same "codimension". In this case, the relevant derivative is the matrix of partial derivatives of $A \mapsto A^T A$, writen in a suitable basis. It follows that
$$\dim O(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$
The dimension statement can also be seen directly as follows. If A is an orthogonal matrix, its first column is just a point of S^{n-1}. Then its second column is a point on the sphere

orthogonal to the first column, so it lives in an "equator", meaning a sphere of dimension one less. Continuing in this way, we see that the "degree of freedom" for specifying a point of $O(n)$ is $(n-1)+(n-2)+\cdots+1 = \frac{n(n-1)}{2}$.

(7) $\mathrm{Gr}_{k,n}(\mathbb{R})$ is a $k(n-k)$-manifold. One way to see this is to use the homeomorphism

$$\mathrm{Gr}_{k,n}(\mathbb{R}) \cong O(n)/\big(O(k)\times O(n-k)\big)$$

from Example 15.7. We get

$$\begin{aligned}\dim \mathrm{Gr}_{n,k}(\mathbb{R}) &= \dim O(n) - \big(\dim O(k) + \dim O(n-k)\big)\\ &= \sum_{j=1}^{n-1} j - \left(\sum_{j=1}^{k-1} j + \sum_{\ell=1}^{n-k-1} \ell\right) = \sum_{j=k}^{n-1} j - \sum_{\ell=1}^{n-k-1} \ell\\ &= \sum_{\ell=0}^{n-k-1} k+\ell - \sum_{\ell=0}^{n-k-1} \ell = \sum_{\ell=0}^{n-k-1} k = k(n-k)\end{aligned}$$

Here are some nonexamples of manifolds.

Example 19.3. (1) The union of the coordinate axes in $\mathbb{R}^2$. Every point has a neighborhood like $\mathbb{R}^1$ except for the origin.

(2) A discrete uncountable set is not second countable.

(3) A 0-manifold is discrete, so $\mathbb{Q}$ is not a 0-manifold.

(4) Glue together two copies of $\mathbb{R}$ by identifying any nonzero x in one copy with the point x in the other. This is second-countable and looks locally like $\mathbb{R}^1$, but it is not Hausdorff.

19.1. **Properties of Manifolds.**

Proposition 19.4. *Any manifold is locally path-connected.*

This follows immediately since a manifold is locally Euclidean.

Proposition 19.5. *Any manifold is normal.*

Proof. This follows from Theorem 18.16. To see that a manifold M is locally compact, consider a point $x \in M$. Then x has a Euclidean neighborhood $x \in U \subseteq M$. U is homeomorphic to an open subset V of $\mathbb{R}^n$, so we can find a compact neighborhood K of x in V (think of a closed ball in $\mathbb{R}^n$). Under the homeomorphism, K corresponds to a compact neighborhood of x in U. ∎

It also follows similarly that any manifold is metrizable, but we can do better. It is convenient to introduce the following term.

19.2. **Embedding.**

Theorem 19.6. *Any manifold M^n admits an embedding into some Euclidean space $\mathbb{R}^N$.*

Sketch. The theorem is true as stated, but we only prove it in the case of a compact manifold. Note that in this case, since M is compact and $\mathbb{R}^N$ is Hausdorff, it is enough to find a continuous injection of M into some $\mathbb{R}^N$.

Since M is a manifold, it has an open cover by sets that are homeomorphic to $\mathbb{R}^n$. Since it is compact, there is a finite subcover $\{U_1,\ldots,U_k\}$. The idea is to then use Urysohn's lemma to extend these homeomorphisms $U_i \cong \mathbb{R}^n$ to functions $f_i : M \longrightarrow \mathbb{R}^n$. Technically, this uses what is called a "partition of unity". Then the collection of functions $\{f_i\}$ give a single function $f : M \longrightarrow (\mathbb{R}^n)^k$. Often, this is an injection, but if the cover is not very well-behaved then it is necessary to also tack on the k Urysohn functions in order to get an injection $M \hookrightarrow \mathbb{R}^{nk+k}$. ∎

In fact, one can do better. Munkres shows (Cor. 50.8) that every compact n-manifold embeds inside $\mathbb{R}^{2n+1}$.