Mon, Nov. 27

Last time, we were discussing CW complexes, and we considered two different CW structures on S^n. We continue with more examples.

(2) (Torus) In general, a product of two CW complexes becomes a CW complex. We will describe this in the case $S^1 \times S^1$, where S^1 is built using a single 0-cell and single 1-cell.

Start with a single 0-cell, and attach two 1-cells. This gives $S^1 \vee S^1$. Now attach a single 2-cell to the 1-skeleton via the attaching map ψ defined as follows. Let us refer to the two circles in $S^1 \vee S^1$ as ℓ and r. We then specify $\psi : S^1 \longrightarrow S^1 \vee S^1$ by $\ell r \ell^{-1} r^{-1}$. What we mean is to trace out ℓ on the first quarter of the domain, to trace out r on the second quarter, to run ℓ in reverse on the third quarter, and finally to run r in reverse on the final quarter.

We claim that the resulting CW complex X is the torus. Since the attaching map $\psi : S^1 \longrightarrow S^1 \vee S^1$ is surjective, so is $\iota_{D^2} : D^2 \longrightarrow X$. Even better, it is a quotient map. On the other hand, we also have a quotient map $I^2 \longrightarrow T^2$, and using the homeomorphism $I^2 \cong D^2$ from before, we can see that the quotient relation in the two cases agrees. We say that this homeomorphism $T^2 \cong X$ puts a cell structure on the torus. There is a single 0-cell (a vertex), two 1-cells (the two circles in $S^1 \vee S^1$), and a single 2-cell.

(3) $\mathbb{RP}^n$. Let's start with $\mathbb{RP}^2$. Recall that one model for this space was as the quotient of D^2, where we imposed the relation $x \sim -x$ on the boundary. If we restrict our attention to the boundary S^1, then the resulting quotient is $\mathbb{RP}^1$, which is again a circle. The quotient map $q : S^1 \longrightarrow S^1$ is the map that winds twice around the circle. In complex coordinates, this would be $z \mapsto z^2$. The above says that we can represent $\mathbb{RP}^2$ as the pushout

$$\begin{tikzcd} S^1 \arrow[r, "\iota"] \arrow[d, "q"'] & D^2 \arrow[d] \\ S^1 \arrow[r] & \mathbb{RP}^2 \end{tikzcd}$$

If we build the 1-skeleton S^1 using a single 0-cell and a single 1-cell, then $\mathbb{RP}^2$ has a single cell in dimensions ≤ 2.

More generally, we can define $\mathbb{RP}^n$ as a quotient of D^n by the relation $x \sim -x$ on the boundary S^{n-1}. This quotient space of the boundary was our original definition of $\mathbb{RP}^{n-1}$. It follows that we can describe $\mathbb{RP}^n$ as the pushout

$$\begin{tikzcd} S^{n-1} \arrow[r, "\iota"] \arrow[d, "q"'] & D^n \arrow[d] \\ \mathbb{RP}^{n-1} \arrow[r] & \mathbb{RP}^n \end{tikzcd}$$

Thus $\mathbb{RP}^n$ can be built as a CW complex with a single cell in each dimension $\leq n$.

(4) $\mathbb{CP}^n$. Recall that $\mathbb{CP}^1 \cong S^2$. We can think of this as having a single 0-cell and a single 2-cell. We defined $\mathbb{CP}^2$ as the quotient of S^3 by an action of S^1 (thought of as $U(1)$). Let $\eta : S^3 \longrightarrow \mathbb{CP}^1$ be the quotient map. What space do we get by attaching a 4-cell to $\mathbb{CP}^1$ by the map η? Well, the map η is a quotient, so the pushout $\mathbb{CP}^1 \cup_\eta D^4$ is a quotient of D^4 by the S^1-action on the boundary.

Wed, Nov. 29

Now include D^4 into $S^5 \subseteq \mathbb{C}^3$ via the map

$$\varphi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4, \sqrt{1 - \sum x_i^2}, 0).$$

(This would be a hemi-equator.) We have the diagonal $U(1)$ action on S^5. But since any nonzero complex number can be rotated onto the positive x-axis, the image of φ meets every S^1-orbit in S^5, and this inclusion induces a homeomorphism on orbit spaces

$$D^4/U(1) \quad \cong S^5/U(1) = \mathbb{CP}^2.$$

We have shown that $\mathbb{CP}^2$ has a cell structure with a single 0-cell, 2-cell, and 4-cell.

This story of course generalizes to show that any $\mathbb{CP}^n$ can be built as a CW complex having a cell in each even dimension.

Let's talk about some of the (nice!) topological properties of CW complexes.

21.3. **Niceness.**

Theorem 21.12 (Hatcher, Prop A.3)**.** *Any CW complex X is normal.*

Even better,

Theorem 21.13 (Lee, Theorem 5.22)**.** *Every CW complex is paracompact.*

Proposition 21.14. *Any CW complex X is locally path-connected.*

Proof. Let $x \in X$ and let U be any open neighborhood of x. We want to find a path-connected neighborhood V of x in U. Recall that a subset $V \subseteq X$ is open if and only if $V \cap X^n$ is open for all n. We will define V by specifying open subsets $V^n \subseteq X^n$ with $V^{n+1} \cap X^n = V^n$ and then setting $V = \cup V^n$.

Suppose that x is contained in the (interior of the) cell e_i^n. We set $V^k = \emptyset$ for $k < n$. We specify V_n by defining $\Phi_j^{-1}(V^n)$ for each n-cell e_j^n. If $j \neq i$, we set $\Phi_j^{-1}(V_n) = \emptyset$. We define $\Phi_i^{-1}(V_n)$ to be an open n-disc around $\Phi_i^{-1}(x)$ whose closure is contained in $\Phi_i^{-1}(U)$. Now suppose we have defined V^k for some $k \geq n$. Again, we define V^{k+1} by defining each $\Phi_j^{-1}(V^{k+1})$. By assumption, $\overline{\Phi_j^{-1}(V^k)} \subseteq \partial D^{k+1} \subseteq \Phi_j^{-1}(U)$. By the Tube lemma, there is an $\epsilon > 0$ such that (using radial coordinates) $\Phi_j^{-1}(V^k) \times (1-\epsilon, 1] \subset \Phi_j^{-1}(U)$. We define

$$\Phi_j^{-1}(V^{k+1}) = \Phi_j^{-1}(V^k) \times [1, 1-\epsilon/2),$$

which is path-connected by induction. Note that this forces $\Phi_j^{-1}(V^{k+1})$ to be empty if the image of the attaching map for the cell e_j^{k+1} does not meet V_k. Now by construction V^{k+1} is the overlapping union of path-connected sets and therefore path-connected. This also guarantees that $\overline{V^{k+1}} \subset U \cap X^{k+1}$, allowing the induction to proceed. ■

Proposition 21.15 (Hatcher, A.1)**.** *Any compact subset K of a CW complex X meets finitely many cells.*

Corollary 21.16. *Any CW complex has the closure-finite property, meaning that the closure of any cell meets finitely many cells.*

Proof. The closure of e_i is $\Phi_i(D_i^{n_i})$, which is compact. The result follows from the proposition. ■

Corollary 21.17.
(i) A CW complex X is compact if and only if it has finitely many cells.
(ii) A CW complex X is locally compact if and only if the collection $\mathcal{E}$ of cells is locally finite.

Part 6. Homotopy and the fundamental group

22. Homotopy

Fri, Dec. 1

We have studied a number of topological properties of spaces, but how would we use these to distinguish S^2, $\mathbb{RP}^2$, and T^2? These are all compact, connected 2-manifolds. It turns out that the fundamental group will allow us to distinguish these spaces. This is the start of **algebraic** topology. We first introduce the idea of a homotopy.

Definition 22.1. Given maps f and $g : X \longrightarrow Y$, a **homotopy** h between f and g is a map $h : X \times I \longrightarrow Y$ ($I = [0, 1]$) such that $f(x) = h(x, 0)$ and $g(x) = h(x, 1)$. We say f and g are **homotopic** if there exists a homotopy between them (and write $h : f \simeq g$).

Example 22.2. Let $f = \mathrm{id} : \mathbb{R} \longrightarrow \mathbb{R}$ and take $g : \mathbb{R} \longrightarrow \mathbb{R}$ to be the constant map $g(x) = 0$. Then a homotopy $h : f \simeq g$ is given by

$$h(x, t) = x(1 - t).$$

Check that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$. Since f is homotopic to a constant map, we say that f is **null-homotopic** (and h is a **null-homotopy**).

Example 22.3. Consider $f = \mathrm{id} : S^1 \longrightarrow S^1$ and the map $g : S^1 \longrightarrow S^1$ defined by $g(\cos(\theta), \sin(\theta)) = (\cos(2\theta), \sin(2\theta))$. Thinking of S^1 as the complex numbers of unit norm, the map g can alternatively be described as $g(z) = z^2$. Then the maps f and g are <u>not</u> homotopic. Furthermore, neither is null-homotopic. (Though we won't be able to show this until next semester.)

Proposition 22.4. *The property of being homotopic defines an equivalence relation on the set of maps $X \longrightarrow Y$.*

Proof. (Reflexive): Need to show $f \simeq f$. Use the **constant homotopy** defined by $h(x, t) = f(x)$ for all t.

(Symmetric): If $h : f \simeq g$, we need a homotopy from g to f. Define $H(x, t) = h(x, 1-t)$ (reverse time).

(Transitive): If $h_1 : f_1 \simeq f_2$ and $h_2 : f_2 \simeq f_3$, we define a new homotopy h from f_1 to f_3 by the formula

$$h(x,t) = \begin{cases} h_1(x, 2t) & 0 \le t \le 1/2 \\ h_2(x, 2t-1) & 1/2 \le t \le 2. \end{cases}$$

■

We write $[X, Y]$ for the set of homotopy classes of maps $X \longrightarrow Y$.

Proposition 22.5. *(Interaction of composition and homotopy) Suppose given maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $X \xrightarrow{f'} Y \xrightarrow{g'} Z$. If $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f'$.*

Proof. We will show that $g \circ f \simeq g' \circ f$. The required homotopy is given by

$$H(x, t) = h'(f(x), t).$$

It is easily verified that $H(x, 0) = g \circ f(x)$ and $H(x, 1) = g' \circ f(x)$. Why is the map $H : X \times I \longrightarrow Z$ continuous? It is the composition of the continuous maps

$$X \times I \xrightarrow{f \times \mathrm{id}} Y \times I \xrightarrow{h'} Z.$$

That the map $f \times \mathrm{id}$ is continuous can be easily verified using the universal property. ■

Definition 22.6. A map $f : X \longrightarrow Y$ is a **homotopy equivalence** if there is a map $g : Y \longrightarrow X$ such that both composites $f \circ g$ and $g \circ f$ are homotopic to the identity maps. We say that spaces X and Y are **homotopy equivalent** if there exists some homotopy equivalence between them, and we write $X \simeq Y$.

Remark 22.7. It is clear that any homeomorphism is a homotopy equivalence, since then both composites are *equal* to the idenitity maps.

The following example shows that the converse is not true.

Example 22.8. The (unique) map $f : \mathbb{R} \longrightarrow *$, where $*$ is the one-point space, is a homotopy equivalence. Pick *any* map $g : * \longrightarrow \mathbb{R}$ (for example, the inclusion of the origin). Then $f \circ g = \mathrm{id}$. The other composition $g \circ f : \mathbb{R} \longrightarrow \mathbb{R}$ is contant, but we have already seen last time that the identity map of $\mathbb{R}$ is null-homotopic. So $\mathbb{R} \simeq *$. The same argument works equally well to show that $\mathbb{R}^n \simeq *$ for any n. Even more generally, if X is a convex subset of $\mathbb{R}^n$, then $X \simeq *$.

Here's some more terminology: any space that is homotopy-equivalent to the one-point space is said to be **contractible**. As we have just seen in the example above, this is equivalent to the statement that the identity map is null-homotopic.

More generally, we can show that any two maps $f, g : X \rightrightarrows \mathbb{R}^n$ are homotopic. The **straight-line homotopy** between f and g is given by

$$h(x, t) = (1 - t)f(x) + tg(t).$$

We will see next semester that the spaces S^2, $\mathbb{RP}^2$, and T^2 are not homotopy-equivalent (and therefore not homeomorphic).