Mon, Aug. 28

2.2. **Convergence.** In calculus, we are also used to thinking of continuity in terms of convergence of sequences. Recall that a sequence (x_n) in X **converges** to x if for every $\epsilon > 0$ there exists N such that for all $n > N$, we have $x_n \in B_\epsilon(x)$. We say that a "tail" of the sequence is contained in the ball around x.

Proposition 2.8. *The sequence (x_n) converges to x if and only if for every open set U containing x, some tail of (x_n) lies in U.*

Proof. Exercise. ■

Proposition 2.9. *Let $f : X \longrightarrow Y$ be a function between metric spaces. The following are equivalent:*

(1) *f is continuous*
(6) *For every convergent sequence $(x_n) \to x$ in X, the sequence $(f(x_n))$ converges to $f(x)$ in Y.*

Proof. This is on HW1. ■

This finishes our discussion of continuity.

What constructions can we make with metric spaces?

3. Products

Let's start with a product. That is, if (X, d_X) and (Y, d_Y) are metric spaces, is there a good notion of the product metric space? We would want to have continuous "projection" maps to each of X and Y, and we would want it to be true that to define a continuous map from some metric space Z to the product, it is enough to specify continuous maps to each of X and Y. By thinking about the case in which Z has a discrete metric, one can see that the underlying set of the product metric space would need to be the cartesian product $X \times Y$. The only question is whether or not there is a sensible metric to define.

Recall that we discussed several metrics on $\mathbb{R}^2$, including the standard one, the max metric, and the taxicab metric. There, we used that $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ as an underlying set, and we combined the metrics on each copy of $\mathbb{R}$ to get a metric on $\mathbb{R}^2$. We can use the same idea here to get three different metrics on $X \times Y$, and these will all produce a metric space satisfying the right property to be a product.

For convenience, let's pick the max metric on $X \times Y$. To show that the projection $p_X : X \times Y \longrightarrow X$ is continuous, it is enough to show that each $p_X^{-1}(B_\epsilon(x))$ is open. But it is simple to show that

$$p_X^{-1}(B_\epsilon(x)) = B_\epsilon(x) \times Y$$

is open using the max metric. The same argument shows that p_Y is continuous.

Now suppose that Z is another metric space with continuous maps $f_X, f_Y : Z \rightrightarrows X, Y$. We define $f = (f_X, f_Y)$ coordinate-wise as before, and it only remains to show that it is continuous. Consider a ball $B_\epsilon(x, y) \subset X \times Y$. Under the max metric, this ball can be rewritten as

$$B_\epsilon(x, y) = B_\epsilon(x) \times B_\epsilon(y),$$

so that

$$f^{-1}(B_\epsilon(x, y)) = f^{-1}\big(B_\epsilon(x) \times B_\epsilon(y)\big) = f_X^{-1}(B_\epsilon(x)) \cap f_Y^{-1}(B_\epsilon(y)).$$

By a problem from HW1, this is open, showing that f is continuous.

Question 3.1. *What if we have infinitely many metric spaces X_i, as i ranges through some indexing set $\mathcal{I}$? Can we make sense of a product $\prod_{i\in\mathcal{I}} X_i$ as a metric space?*

Wed, Aug. 30

Last time, we discussed the product of metric spaces. We were guided to our definition by considering the "universal property" that the product should satisfy. But why is this enough? Couldn't there be another metric space that shares the same universal property? It turns out that the answer is no: any two metric spaces that share the same universal property are essentially "the same". We will come back to this point when we discuss products in the context of topological spaces.

4. Function spaces

Another important construction is that of a space of functions. That is, if X and Y are metric spaces, one can consider the set of all continuous functions $f : X \longrightarrow Y$. Is there a good way to think of this as a metric space? For example, as a set $\mathbb{R}^2$ is the same as the collection of functions $\{1,2\} \longrightarrow \mathbb{R}$. More generally, we could consider functions $\{1,\ldots,n\} \longrightarrow Y$ or even $\mathbb{N} \longrightarrow Y$ (i.e. sequences).

Again, we first step back and consider what happens in the simpler world of sets. We will write $F(Y,Z)$ for the set of functions $Y \longrightarrow Z$, also denoted Z^Y. Then given a function $\varphi : X \times Y \longrightarrow Z$, we can define a function $\hat{\varphi} : X \longrightarrow F(Y,Z)$ by the formula

$$\hat{\varphi}(x)(y) = \varphi(x,y).$$

Conversely, given a function $\hat{\varphi}$, the equation above defines the function φ. In other words, we have a bijection

$$F(X \times Y, Z) \cong F(X, F(Y,Z)).$$

We might ask if a similar story exists in the world of metric spaces.

Of the metrics we discussed on $\mathbb{R}^2$, the max metric generalizes most easily to give a metric on $Y^\infty = Y^{\mathbb{N}}$. We provisionally define the **sup metric** on the set of sequences in Y by

$$d_{\sup}((y_n),(z_n)) = \sup_n \{d_Y(y_n, z_n)\}.$$

Without any further restrictions, there is no reason that this supremum should always exist. If Y is a bounded metric space, or if we only consider bounded sequences, then we are OK. Another option is to arbitrarily truncate the metric.

Lemma 4.1. *Let (Y,d) be a metric space. Define the resulting bounded metric $\overline{d}$ on Y by*

$$\overline{d}(y,z) = \min\{d(y,z), 1\}.$$

This is a metric, and the open sets determined by $\overline{d}$ are precisely the open sets determined by d.

We now redefine the sup metric on Y^∞ to be

$$d_{\sup}((y_n),(z_n)) = \sup_n \{\overline{d_Y}(y_n, z_n)\}.$$

Now the supremum always exists, so that we get a well-defined metric. The same definition works to give a metric on the set of continuous functions $X \longrightarrow Y$. We define the sup metric on the set $\mathcal{C}(X,Y)$ of continuous functions to be

$$d_{\sup}(f,g) = \sup_{x\in X} \{\overline{d_Y}(f(x), g(x))\}.$$

This is also called the **uniform metric**, for the following reason.

Proposition 4.2. *Let (f_n) be a sequence in $\mathcal{C}(X,Y)$. Then $(f_n) \to f$ in the uniform metric on $\mathcal{C}(X,Y)$ if and only if $(f_n) \to f$ uniformly.*

Given a function $f \in \mathcal{C}(X,Y)$ and a point $x \in X$, one can evaluate the function to get $f(x) \in Y$. In other words, we have an evaluation function

$$\mathrm{eval} : \mathcal{C}(X,Y) \times X \longrightarrow Y.$$

Proposition 4.3. *Consider $\mathcal{C}(X,Y) \times X$ as a metric space using the max metric. Then* eval *is continuous.*

Proof. By Proposition **??**, to determine if a function between metric spaces is continuous, it suffices to check that it takes convergent sequences to convergent sequences. Suppose that $(f_n, x_n) \to (f, x)$. We wish to show that

$$\mathrm{eval}(f_n, x_n) = f_n(x_n) \to \mathrm{eval}(f, x) = f(x).$$

Since $(f_n, x_n) \to (f, x)$, it follows that $f_n \to f$ and $x_n \to x$ (since the projections are continuous).

Let $\varepsilon > 0$. Then there exists N_1 such that if $n > N_1$ then $d_{\sup}(f_n, f) < \varepsilon/2$. By the definition of the sup metric, this implies that $d_Y(f_n(x_n), f(x_n)) < \varepsilon/2$. But now f is continuous, so there exists N_2 such that if $n > N_2$ then $d_Y(f(x_n), f(x)) < \varepsilon/2$. Putting these together and using the triangle inequality, if $n > N_3 = \max\{N_1, N_2\}$ then $d_Y(f_n(x_n), f(x)) < \varepsilon$. ■

$\boxed{\textbf{Fri, Sept. 1}}$

Proposition 4.4. *Suppose $\varphi : X \times Y \longrightarrow Z$ is continuous. For each $x \in X$, define $\hat{\varphi}(x) : Y \longrightarrow Z$ by $\hat{\varphi}(x)(y) = \varphi(x, y)$. The function $\hat{\varphi}(x)$ is continuous.*

Proof. This could certainly be done directly, using convergence of sequences to test for continuity. Here is another way to do it, using the universal property of products.

Note that $\hat{\varphi}(x)$ can be written as the composition $Y \xrightarrow{i_x} X \times Y \xrightarrow{\varphi} Z$. By assumption, φ is continuous, so it suffices to know that $i_x : Y \to X \times Y$ is continuous. But recall that continuous maps into a product correspond precisely to a pair of continuous maps into each factor. The pair of maps here is the constant map $Y \longrightarrow X$ at x and the identity map $Y \longrightarrow Y$. The identity map is clearly continuous, and the constant map is continuous since if $U \subseteq X$ is open, then the preimage under the constant map is either (1) all of Y if $x \in U$ or (2) empty if $x \notin U$. So it follows that i_x is continuous. ■

We are headed to the universal property of the mapping space. Keeping the notation from above, given a continuous function

$$\varphi : X \times Y \longrightarrow Z,$$

we get a function

$$\hat{\varphi} : X \longrightarrow \mathcal{C}(Y, Z).$$

Conversely, given the function $\hat{\varphi}$, we define φ by

$$\varphi(x, y) = \hat{\varphi}(x)(y).$$

Proposition 4.5. *The function φ above is continuous if $\hat{\varphi}$ is continuous.*

Proof. On homework 2. ■

Of course, we would like this to be an if and only if, but that is only true under additional hypotheses (like Y compact, for instance.) Another way to state the if-and-only-if version of this proposition is that we get a bijection

$$\mathcal{C}(X \times Y, Z) \cong \mathcal{C}(X, \mathcal{C}(Y, Z)).$$

For those who have seen the $(\otimes, \mathrm{Hom})$ adjunction in algebra, this is completely analogous.

5. Quotients

Another (very) important construction that we will discuss when we move on to topological spaces is that of a quotient, or identification space. A standard example is the identification, on the unit interval $[0,1]$, of the two endpoints. Glueing these together gives a circle S^1, and the surjective continuous map

$$e^{2\pi ix} : [0,1] \longrightarrow S^1$$

is called the quotient map. Here the desired universal property is that if $f : [0,1] \longrightarrow Y$ is a continuous map to another metric space such that $f(0) = f(1)$, then the map f should "factor" through the quotient. Quotients become quite complicated to express in the world of metric spaces.

Part 2. Topological Spaces

Now that we have spent some time with metric spaces, let's turn to the more general world of topological spaces.

6. Definitions

Definition 6.1. A **topological space** is a set X with a collection of subsets $\mathcal{T}$ of X such that

(1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$
(2) If $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$
(3) If $U_i \in \mathcal{T}$ for all i in some index set I, then $\bigcup_{i\in I} U_i \in \mathcal{T}$.

The collection $\mathcal{T}$ is called the **topology** on X, and the elements of $\mathcal{T}$ are referred to as the "open sets" in the topology.

Example 6.2.

(1) (Metric topology) Any metric space is a topological space, where $\mathcal{T}$ is the collection of metric open sets
(2) (Discrete topology) In the discrete topology, *every* subset is open. We already saw the discrete metric on any set, so in fact this is an example of a metric topology as well.
(3) (Trivial topology) In the trivial topology, $\mathcal{T} = \{\emptyset, X\}$. That is, $\emptyset$ and X are the only empty sets. This topology does not come from a metric (unless X has fewer than two points).
(4) It is simple to write down various topologies on a finite set. For example, on the set

$$X = \{1,2\},$$

there are 4 possible topologies. In addition to the trivial and discrete topologies, there is also

$$\mathcal{T}_1 = \{\emptyset, \{1\}, X\}$$

and

$$\mathcal{T}_2 = \{\emptyset, \{2\}, X\}.$$

(5) There are many possible topologies on $X = \{1,2,3\}$. But not every collection of subsets will give a topology. For instance,

$$\{\emptyset, \{1,2\}, \{1,3\}, X\}$$

would not be a topology, since it is not closed under intersection.