Mon, Sept. 11

9. Closed Sets

So far, we only discussed the notion of open set, but there is also the complementary notion of closed set.

Definition 9.1. Let X be a space. We say a subset $W \subseteq X$ is **closed** if the complement $X \setminus W$ is open.

Note that, despite what the name may suggest, closed does *not* mean "not open". For instance, the empty set is always both open (required for any topology) and closed (because the complement, X must be open). Similarly, there are many examples of sets that are neither open nor closed (for example, the interval $[0, 1)$ in the usual topology on $\mathbb{R}$).

Proposition 9.2. *Let X be a space.*

(1) *$\emptyset$ and X are both closed in X*
(2) *If W_1, W_2 are closed, then $W_1 \cup W_2$ is also closed*
(3) *If W_i are closed for all i in some index set I, then $\bigcap_{i \in I} W_i$ is also closed.*

Proof. We prove (2). The point is that

$$X \setminus (W_1 \cup W_2) = (X \setminus W_1) \cap (X \setminus W_2).$$

This equality is known as one of the DeMorgan Laws ■

Example 9.3. Consider $\mathbb{R}_{\ell\ell}$, the real line equipped with the lower-limit topology. (Example 6.6). There, a half-open interval $[a, b)$ was declared to be open. It then follows that intervals of the form $(-\infty, b)$ and $[a, \infty)$ are open. But this then implies that $[a, b)$ is *closed* since its complement is the open set $(-\infty, a) \cup [b, \infty)$.

Not only does a topology give rise to a collection of closed sets satisfying the above properties, but one can also define a topology by specifying a list of closed sets satisfying the above properties.

Similarly, we can use closed sets to determine continuity.

Proposition 9.4. *Let $f : X \longrightarrow Y$. Then f is continuous if and only if the preimage of every closed set in Y is closed in X.*

Example 9.5. The "distance from the origin function" $d : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is continuous (follows from HW 2). Since $\{1\} \subseteq \mathbb{R}$ is closed, it follows that the sphere $S^2 = d^{-1}(1)$ is closed in $\mathbb{R}^3$. More generally, S^{n-1} is closed in $\mathbb{R}^n$.

Example 9.6. Let X be any metric space, let $x \in X$, and let $r > 0$. Then the ball

$$B_{\leq r}(x) = \{y \in X \mid d(x, y) \leq r\}$$

is closed in X.

Remark 9.7. Note that some authors use the notation $\overline{B_r(x)}$ for the closed ball. This is a bad choice of notation, since it suggests that the closure of the open ball is the closed ball. But this is not always true! For instance, consider a set (with more than one point) equipped with the discrete metric. Then $B_1(x) = \{x\}$ is already closed, so it is its own closure. On the other hand, $B_{\leq 1}(x) = X$.

Consider the half-open interval $[a, b)$. It is neither open nor closed, in the usual topology. Nevertheless, there is a closely associated closed set, $[a, b]$. Similarly, there is a closely associated open set, (a, b). Notice the containments

$$(a, b) \subseteq [a, b) \subseteq [a, b].$$

It turns out that this picture generalizes.

9.1. **Closure and Interior.** Let's start with the closed set. In the example above, $[a, b]$ is the *smallest closed set containing* $[a, b)$. Why should we expect such a smallest closed set to exist in general? Recall that if we intersect arbitrarily many closed sets, we are left with a closed set.

Definition 9.8. Let $A \subseteq X$ be a subset of a topological space. We define the **closure of A in X** to be

$$\overline{A} = \bigcap_{A \subset B \text{ closed}} B.$$

Dually, we have $(a, b) \subset [a, b)$, and (a, b) is the largest open set contained in $[a, b)$.

Definition 9.9. Let $A \subseteq X$ be a subset of a topological space. We define the **interior of A in X** to be

$$\mathrm{Int}(A) = \bigcup_{A \supset U \text{ open}} U.$$

The difference of these two constructions is called the **boundary of A in X**, defined as

$$\partial A = \overline{A} \setminus \mathrm{Int}(A).$$

Example 9.10. (1) From what we have already said, it follows that $\partial[a, b) = \{a, b\}$.

(2) Let $A = \{1/n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$. Then A is not open, since no neighborhood of any $1/n$ is contained in A. This also shows that $\mathrm{Int}(A) = \emptyset$. But neither is A closed, because no neighborhood of 0 is contained in the complement of A. This implies that $0 \in \overline{A}$, and it turns out that $\overline{A} = A \cup \{0\}$. Thus $\partial A = \overline{A} = A \cup \{0\}$.

(3) Let $\mathbb{Q} \subseteq \mathbb{R}$. Similarly to the example above, $\mathrm{Int}(\mathbb{Q}) = \emptyset$. But since $\mathbb{R} \setminus \mathbb{Q}$ does not entirely contain any open intervals, it follows that $\overline{\mathbb{Q}} = \mathbb{R}$. (A subset $A \subseteq X$ is said to be **dense** in X if $\overline{A} = X$.) Thus $\partial\mathbb{Q} = \mathbb{R} \setminus \emptyset = \mathbb{R}$.

(4) Let's turn again to $\mathbb{R}_{\ell\ell}$. We saw that $[0, 1)$ was already closed. What about $(0, 1]$? Since $[0, 1]$ is closed in the usual topology, this must be closed in $\mathbb{R}_{\ell\ell}$ as well. (Recall that the topology on $\mathbb{R}_{\ell\ell}$ is finer than the standard one). It follows that $(0, 1]$ is either already closed, or its closure is $[0, 1]$. We can ask, dually, whether the complement is open. But $(-\infty, 0] \cup (1, \infty)$ is not open since it does not contain any neighborhoods of 0. It follows that $\overline{(0, 1]} = [0, 1]$ in $\mathbb{R}_{\ell\ell}$.

There is a convenient characterization of the closure, which we were implicitly using above.

Proposition 9.11 (Neighborhood criterion)**.** *Let $A \subseteq X$. Then $x \in \overline{A}$ if and only if every neighborhood of x meets A.*

Proof. ($\Rightarrow$) Suppose $x \in \overline{A}$. Then $x \in B$ for all closed sets B containing A. Let N be a neighborhood of x. Without loss of generality, we may suppose N is open. Now $X \setminus N$ is closed but $x \notin X \setminus N$, so this set cannot contain A. This means precisely that $N \cap A \neq \emptyset$.

($\Leftarrow$) Suppose every neighborhood of x meets A. Let $A \subset B$, where B is closed in X. Now $U = X \setminus B$ is an open set not meeting A, so it cannot be a neighborhood of x. This must mean that $x \notin X \setminus B$, or in other words $x \in B$. Since B was arbitrary, it follows that x lies in every such B. ■

Wed, Sept. 13

10. Convergence

In our earlier discussion of metric spaces, we considered convergence of sequences and how this characterized continuity. This is one statement from the theory of metric spaces that will not carry over to the generality of topological spaces.

Definition 10.1. We say that a sequence x_n in X converges to x in X if every neighborhood of x contains a tail of (x_n).

The following result follows immediately from the previous characterization of the closure.

Proposition 10.2. *Let (a_n) be a sequence in $A \subseteq X$ and suppose that $a_n \to x \in X$. Then $x \in \overline{A}$.*

Proof. We use the neighborhood criterion. Thus let U be a neighborhood of x. Since $a_n \to x$, a tail of (a_n) lies in U. It follows that $U \cap A \neq \emptyset$, so that $x \in \overline{A}$. ∎

However, the converse is not true in a general topological space. (The fact that these are equivalent in a metric space is the **sequence lemma**, Proposition 10.12.)

Example 10.3. Consider $\mathbb{R}$ equipped with the *cocountable* topology. Recall that this means that the nonempty open subsets are the cocountable ones.

Lemma 10.4. *Suppose that $x_n \to x$ in the cocountable topology on $\mathbb{R}$. Then (x_n) is eventually constant.*

Proof. Write B for the set

$$B = \{x_n \mid x_n \neq x\}.$$

Certainly B is countable, so it is closed. By construction, $x \notin B$, so $N = X \setminus B$ is an open neighborhood of x. But $x_n \to x$, so a tail of this sequence must lie in N. Since $\{x_n\} \cap N = \{x\}$, this means that a tail of this sequence is constant, in other words, the sequence is eventually constant. ∎

Now consider $A = \mathbb{R} \setminus \{0\} \subseteq \mathbb{R}$ in the cocountable topology. A is not closed since the only closed proper subsets are the countable ones. It follows that A must be dense in $\mathbb{R}$. However, no sequence in A can converge to 0 since a convergent sequence must be eventually constant.

Similarly, we cannot use convergence of sequences to test for continuity in general topological spaces. For instance, consider the identity map

$$\mathrm{id} : \mathbb{R}_{\text{cocountable}} \longrightarrow \mathbb{R}_{\text{standard}},$$

where the domain is given the cocountable topology and the codomain is given the usual topology. This is not continuous, since the interval $(0, 1)$ is open in $\mathbb{R}_{\text{standard}}$ but not in $\mathbb{R}_{\text{cocountable}}$. On the other hand, the identity function takes convergent sequences in $\mathbb{R}_{\text{cocountable}}$, which are necessarily eventually constant, to convergent sequences in $\mathbb{R}_{\text{standard}}$. This follows from the following result, which you proved on HW1.

Proposition 10.5. *Let $f : X \longrightarrow Y$ be continuous. If $x_n \to x$ in X then $f(x_n) \to f(x)$ in Y.*

Proof. Suppose $x_n \to x$. Let V be an open neighborhood of $f(x)$. Then, since f is continuous, $f^{-1}(V)$ is an open neighborhood of x. Thus some tail of (x_n) lies in $f^{-1}(V)$, which means that the corresponding tail of $(f(x_n))$ lies in U. ∎

However, all hope is not lost, since the following is true.

Proposition 10.6. *Let $f : X \longrightarrow Y$. Then f is continuous if and only if*

$$f(\overline{A}) \subseteq \overline{f(A)}$$

for every subset $A \subseteq X$.

Proof. ($\Rightarrow$) Assume f is continuous. Since $\overline{f(A)}$ is the intersection of *all* closed sets containing $f(A)$, it suffices to show that if B is such a closed set, then $f(\overline{A}) \subseteq B$. Well, $f(A) \subseteq B$, so

$$A = f^{-1}(f(A)) \subseteq f^{-1}(B).$$

Now f is continuous and B is closed, so by definition of the closure, we must have

$$\overline{A} \subseteq f^{-1}(B).$$

Applying f then gives $f(\overline{A}) \subseteq f(f^{-1}(B)) \subseteq B$.

($\Leftarrow$) Suppose that the above subset inclusion holds, and let $B \subseteq Y$ be closed. Let $A = f^{-1}(B)$. We wish to show that A is closed, i.e. that $\overline{A} = A$. Since $f(f^{-1}(B)) \subseteq B$, we know that

$$f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B} = B.$$

Applying f^{-1} gives

$$\overline{A} = f^{-1}(f(\overline{A})) \subseteq f^{-1}(B) = A.$$

It follows that A is closed. ∎

Fri, Sept. 15

10.1. **Accumulation Points.** Ok, so we have learned that points in $\overline{A}$ are good enough to determine continuity of functions, but these points are not necessarily limits of sequences in A. It turns out that there is an alternative characterization of these points.

Definition 10.7. Let X be a space and $A \subseteq X$. A point $x \in X$ is said to be an **accumulation point** (or **cluster point** or **limit point**) of A if

every neighborhood of x contains a point of A other than x itself.

Let us write $\mathrm{acc}(A)$ for the set of accumulation points of A.

Example 10.8. (1) Let $A = (0, 1) \subseteq \mathbb{R}$. Then $\mathrm{acc}(A) = [0, 1]$.
(2) Let $A = \{0, 1\} \subseteq \mathbb{R}$. Then $\mathrm{acc}(A) = \emptyset$.
(3) Let $A = [0, 1) \cup \{2\}$. Then $\mathrm{acc}(A) = [0, 1]$.
(4) Let $A = \{1/n\} \subseteq \mathbb{R}$. Then $\mathrm{acc}(A) = \{0\}$.

The following result follows immediately from our neighborhood characterization of the closure of a set.

Proposition 10.9. *A point x is an acc. point of A if and only if $x \in \overline{A \setminus \{x\}}$.*

Certainly $A \setminus \{x\} \subseteq A$, and the closure operation preserves containment, so it follows that $\mathrm{acc}(A) \subseteq \overline{A}$. From the previous examples, we see that this need not be an equality. We also have $A \subseteq \overline{A}$, and it follows that

$$A \cup \mathrm{acc}(A) \subseteq \overline{A}.$$

Proposition 10.10. *For any subset $A \subseteq X$, we have*

$$A \cup \mathrm{acc}(A) = \overline{A}.$$

Proof. It remains to show that every point in the closure is either in A or in $\mathrm{acc}(A)$. Let $x \in \overline{A}$, but suppose that $x \notin A$. Then, by the neighborhood criterion, we have that for every neighborhood N of x, $N \cap A \neq \emptyset$. But since $x \notin A$, it follows that $N \cap (A \setminus \{x\}) \neq \emptyset$. In other words, $x \in \mathrm{acc}(A)$. ∎

Note that, although the motivation came from looking at sequences, there is no direct relation between accumulation points of A and limits of sequences in A.

We already saw an example of a point in the closure which is not the limit of a sequence. On the other hand, we can ask

Question 10.11. *If (a_n) is a sequence in A and $a_n \to x$, is $x \in \operatorname{acc}(A)$?*

Answer. No. Take $A = \{x\}$ and $a_n = x$. But, if we require that $x \notin A$, then the answer is yes.

As the example $X = \mathbb{R}^n$ suggests, sequences and closed sets are much better behaved for metric spaces.

Proposition 10.12 (The sequence lemma)**.** *Let $A \subseteq X$ and suppose that X is a metric space. Then $x \in \overline{A}$ if and only if x is the limit of a sequence in A.*

Proof. Let $S = \{1/n\}_{n\in\mathbb{N}} \cup \{0\}$, given the subspace topology from $\mathbb{R}$. Then a convergent sequence in a topological space is precisely the same as a continuous map from S to that topological space. We will also write $S_{>0} = S \setminus \{0\}$.

Suppose $a_n \to x$. Then this sequence gives a continuous map $a : S \longrightarrow X$ such that $a(S_{>0}) \subseteq A$. By Proposition 10.6, we know that

$$\operatorname{im}(a) = a(S) = a(\overline{S_{>0}}) \subseteq \overline{A}.$$

In particular, $x = a(0) \in \overline{A}$. This part of the argument does not require X to be metric.

On the other hand, suppose $x \in \overline{A}$. For each n, $B_{1/n}(x)$ is a neighborhood of x, and $x \in \overline{A}$, so $B_{1/n}(x) \cap A \neq \emptyset$. Let $a_n \in B_{1/n}(x) \cap A$. Then the sequence $a_n \to x$, and $a_n \in A$ by construction. ■

Note that by Example 10.3, it follows that the cocountable topology on $\mathbb{R}$ does not come from a metric on $\mathbb{R}$.

10.2. **Countability.** The last few lectures, we have seen that closed sets are not as easily understood in general as they are in the case of metric spaces. Although we will not want to restrict ourselves to metric spaces, it will nevertheless be helpful to have some good characterizations of the "reasonable" spaces. We mention here a few of these properties.

One property of metric spaces that we used recently was the existence of the balls of radius $1/n$.

Definition 10.13. A space X is **first-countable** if, for each $x \in X$, there is a countable collection $\{U_n\}$ of neighborhoods of x such that any other neighborhood contains at least one of the U_n.

This was the key property used in proving that, in a metric space, an accumulation point of $A \subseteq X$ is the limit of an A-sequence. Thus, we have

Proposition 10.14. *Let $f : X \longrightarrow Y$ be a function, where X is first-countable. Then f is continuous if and only if f takes convergent sequences in X to convergent sequences in Y.*

We will return to first-countable (and second-countable) spaces later in the course.

Example 10.15. Again, Proposition 10.14 implies that $X = \mathbb{R}_{\text{cocountable}}$ is not first-countable. We can see this directly as follows. Let $x \in X$ and suppose that $\{U_n\}$ is a collection of neighborhoods of x. By definition, each U_n is open and misses only countably many real numbers. Write $C_n = \mathbb{R} \setminus U_n$. Then $C = \bigcup_n C_n$ is also countable and is therefore a proper subset of $\mathbb{R}$. Let $z \neq x$ be some point in the complement of C. Then $C_2 = C \cup \{z\}$ is countable and strictly contains each C_n. Then $U_2 = X \setminus C_2$ is a neighborhood of x which is strictly contained in each U_n. Thus X is not first-countable.