$\boxed{\textbf{Mon, Sept. 18}}$

10.3. **Hausdorff Spaces.** Another important property of metric spaces is the Hausdorff property.

Definition 10.16. A space X is said to be Hausdorff (also called T_2) if, given any two points x and y in X, there are disjoint open sets U and V with $x \in U$ and $y \in V$.

This is a somewhat mild "separation property" that is held by many spaces in practice and that also has a number of nice consequences.

The Hausdorff property forces sequences to behave well, in the following sense.

Proposition 10.17. *In a Hausdorff space, a sequence cannot converge simultaneously to more than one point.*

Proof. Suppose $x_n \to x$ and $x_n \to y$. Every neighborhood of x contains a tail of x_n, as does any neighborhood of y. It follows that no neighborhood of x is disjoint from any neighborhood of y. Since X is Hausdorff, this forces $x = y$. ■

Proposition 10.18. *Every metric space is Hausdorff.*

Proof. If $x \neq y$, let $d = d(x, y) > 0$. Then the balls of radius $d/2$ centered at x and y are the needed disjoint neighborhoods. ■

However, of the (many, many) topologies on a finite set, the only one that is Hausdorff is the discrete topology. Indeed, if points are closed, then every subset is closed, as it is a finite union of points.

Here is one more nice consequence of this property.

Proposition 10.19. *If X is Hausdorff, then points are closed in X. (A space is called T_1 if points are closed.)*

Proof. The neighborhood crtierion for the complement $X \setminus \{x\}$ is easy to verify. ■

11. Gluing Lemma

In Calculus, you saw functions defined piecewise, and one-sided limits were typically employed to establish continuity. There is an analogue of this type of construction for spaces.

Lemma 11.1 (Glueing/Pasting Lemma)**.** *Let $X = A \cup B$, where either (1) both A and B are open in X or (2) both A and B are closed in X. Then a function $f : X \longrightarrow Y$ is continuous if and only if the restrictions $f_{|A}$ and $f_{|B}$ are both continuous.*

Proof. ($\Rightarrow$) We already proved this in Proposition 8.4.

($\Leftarrow$) We give the proof assuming they are both open. Let $V \subseteq Y$ be open. We wish to show that $f^{-1}(V) \subseteq X$ is open. Let's restrict to A. We have $f^{-1}(V) \cap A = f_{|A}^{-1}(V)$. Since $f_{|A}$ is continuous, it follows that $f_{|A}^{-1}(V)$ is open (in A). Since A is open in X, it follows that $f_{|A}^{-1}(V)$ is also open in X. The same argument shows that $f^{-1}(V) \cap B$ is open in X. It follows that their unoin, which is $f^{-1}(V)$, is open in X. ■

Example 11.2. For example, we can use this to paste together the continuous absolute value function $f(x) = |x|$, as a function $\mathbb{R} \longrightarrow \mathbb{R}$. We get this by pasting the continuous functions $\iota : [0, \infty) \longrightarrow \mathbb{R}$, $x \mapsto x$, and $(-\infty, 0] \cong [0, \infty) \longrightarrow \mathbb{R}$, $x \mapsto -x$.

Example 11.3. Let's look at an example of a discontinuous function, for example

$$f(x) = \begin{cases} 1 & x \neq 1 \\ 2 & x = 1. \end{cases}$$

We can get this by pasting together two constant functions, but the domains are $\mathbb{R} \setminus \{1\}$ and $\{1\}$, one of which is open but not closed, and the other of which is closed but not open.

Example 11.4. Let $X = [0,1] \cup [2,3]$, given the subspace topology from $\mathbb{R}$. Note that in this case each of the subsets $A = [0,1]$ and $B = [2,3]$ is **both** open and closed, so we can specify a continuous function on X by giving a pair of continuous functions, one on A and the other on B.

11.1. **Homeomorphisms.** Finally, we start to look at the idea of sameness. Two sets are thought of as the same if there is a bijection between them. A bijection is simply an invertible function. More generally, we have the following idea.

Definition 11.5. A "morphism" $f : X \longrightarrow Y$ is said to be an **isomorphism** if there is a $g : Y \longrightarrow X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Again, an isomorphism between sets is simply a bijection. In topology, this is called a **homeomorphism**. In other words, a homeomorphism is a continuous function with a continuous inverse. Since such a map is invertible, clearly it must be one-to-one and onto, but it is **not** true that every continuous bijection is a homeomorphism. Before we look at some examples, let's look at some non-examples.

Example 11.6.
(1) Any time a set is equipped with two topologies, one of which is a refinement of the other, the identity map is a continuous bijection (in one direction) that is not a homeomorphism. For instance, we have the following such examples

$$\mathrm{id} : \mathbb{R} \longrightarrow \mathbb{R}_{\text{cofinite}}, \qquad \mathrm{id} : \mathbb{R}_{\text{cocountable}} \longrightarrow \mathbb{R}_{\text{cofinite}} \qquad \mathrm{id} : \mathbb{R}_{\text{discrete}} \longrightarrow \mathbb{R}$$

(2) Consider the exponential map $\exp : [0,1) \longrightarrow S^1$ given by $\exp(x) = e^{2\pi i x}$. This is a continuous bijection, but it is not a homeomorphism. Since homeomorphisms have continuous inverses, they must take open sets to open sets and closed sets to closed sets. But we see that exp does not take the open set $U = [0, 1/2)$ to an open set in S^1. The point $\exp(0) = (1,0)$ has no neighborhood that is contained in $\exp(U)$.

Example 11.7.
(1) Consider $\tan : (0, \frac{\pi}{2}) \longrightarrow (0, \infty)$. This is a continuous bijection with continuous inverse (given by arctangent)

(2) Consider $\ln : (0,\infty) \longrightarrow \mathbb{R}$. This is a continuous bijection with inverse e^x. Composing homeomorphisms produces homeomorphisms, and we therefore get a homeomorphism

$$(0,1) \xrightarrow{\cong} (0, \frac{\pi}{2}) \xrightarrow{\cong} (0,\infty) \xrightarrow{\cong} \mathbb{R}.$$

(3) We similarly get a homeomorphism $\tan : [0, \frac{\pi}{2}) \xrightarrow{\cong} [0,\infty)$. It follows that we have

$$[0,1) \cong [0,\infty) \qquad \text{and} \qquad (0,1] \cong [0,\infty).$$

(4) One can similarly get $B^n_r(x) \cong \mathbb{R}^n$ for any n, r, and x.

The above example shows that there really are only three intervals, up to homeomorphism: the open interval, the half-open interval, and the closed interval.

We say that two spaces are **homeomorphic** if there is a homeomorphism between them (and write $X \cong Y$ as above). This is the notion of "sameness" for spaces. One of the major overarching questions for this course will be: how can we tell when two spaces are the same or are actually different?

A standard way to show that two spaces are not homeomorphic is to find a property that one has and the other does not. For instance every metric space is Hausdorff, so no non-Hausdorff space is the "same" as a metric space. But what property distinguishes the 3 interval types above? As we learn about more and more properties of spaces, this question will become easier to answer.

In the exponential example from last time, we noted that homeomorphisms must take open sets to open sets. Such a map is called an **open map**. Similarly, a **closed map** takes closed sets to closed sets.

Proposition 11.8. *Let $f : X \longrightarrow Y$ be a continuous bijection. The following are equivalent:*

(1) *f is a homeomorphism*
(2) *f is an open map*
(3) *f is a closed map*

If we drop the assumption that f is bijective, it is no longer true that being an open map is equivalent to being a closed map. For example, the inclusion $(0, 1) \longrightarrow \mathbb{R}$ is open but not closed, and the inclusion $[0, 1] \longrightarrow \mathbb{R}$ is closed but not open.

Fri, Sept. 22

Part 3. Constructions

12. Products

Put on your hard hats! We turn now to the construction phase. In section 3, we considered the product of metric spaces: let's define the product for topological spaces. We already know what property it should satisfy: we want it to be true that mapping continuously from some space Z into the product $X \times Y$ should be the same as mapping separately to X and to Y. Another way to describe this is that we want $X \times Y$ to be the "universal" example of a space with a pairs of maps to X and Y.

Well, if the projection $p_X : X \times Y \longrightarrow X$ is to be continuous, we need $p_X^{-1}(U) = U \times Y$ to be open whenever $U \subseteq X$ is open. Similarly, we need $X \times V$ to be open if $V \subseteq Y$ is open. We are forced to include these open sets, but we don't want to throw in anything extra that we don't need. In other words, we want the product topology on $X \times Y$ to be the coarsest topology containing the sets $U \times Y$ and $X \times V$.

Note that if we consider the collecion

$$\mathcal{B} = \{U \times Y \mid U \subseteq X \text{ open}\} \cup \{X \times V \mid V \subseteq Y \text{ open}\},$$

this cannot be a basis because it fails the intersection property. A typical intersection is

$$(U \times Y) \cap (X \times V) = U \times V,$$

and if we consider all sets of this form, we do get a basis.

Definition 12.1. Given spaces X and Y, the **product topology** on $X \times Y$ has basis given by sets of the form $U \times V$, where U and V are open in X and Y, respectively.

This satisfies the universal property of a product. We have engineered the definition to make this so, but we will check this anyway. First, we make a little detour.

We pointed out above that if we considered the collection

$$\mathcal{B} = \{U \times Y\} \cup \{X \times V\},$$

we would not have a basis, as the intersection property failed. We remedied this by considering instead intersections of elements of $\mathcal{B}$. This is a useful idea that shows up often.

Given a set X, a collection $\mathcal{C}$ of subsets of X is called a **prebasis** for a topology on X if the collection covers X. Actually, in all of the textbooks, this is called a subbasis, but that is a terrible name, since it suggests that it is a basis. I will try to stick with the better name of prebasis.

We can then get a basis from the prebasis by considering finite intersections of prebasis elements.

Example 12.2. The collection of rays (a, ∞) and $(-\infty, b)$ give a prebasis for the standard topology on $\mathbb{R}$.

We introduced the product topology above and mentioned the universal property, but let's spend a little bit of time with it to really nail down the concept.

Theorem-Definition 12.3. *Let X and Y be spaces. Then $X \times Y$, together with the projection maps*

$$p_X : X \times Y \longrightarrow X \quad \text{and} \quad p_Y : X \times Y \longrightarrow Y,$$

satisfies the following "universal property": given any space Z and maps $g : Z \longrightarrow X$ and $h : Z \longrightarrow Y$, there is a unique continuous map $f : Z \longrightarrow X \times Y$ such that

$$g = p_X \circ f, \qquad h = p_Y \circ f.$$

$$\begin{tikzcd}
& & X \\
Z \arrow[r, dashed, "\exists! f"] \arrow[urr, bend left, "g"] \arrow[drr, bend right, "h"'] & X \times Y \arrow[ur, "p_X"'] \arrow[dr, "p_Y"] & \\
& & Y
\end{tikzcd}$$

Proof. The uniqueness is clear: if there exists such a continuous map f, then the conditions force this to be $f = (g, h)$. The only question is whether or not f is continuous. Consider a typical basis element $U \times V$ for the product topology on $X \times Y$. Then

$$\begin{aligned} f^{-1}(U \times V) = \{z \in Z \mid f(z) \in U \times V\} &= \{z \in Z \mid g(z) \in U \text{ and } h(z) \in V\} \\ &= g^{-1}(U) \cap h^{-1}(V), \end{aligned}$$

which is an intersection of open sets and therefore open. ∎

Ok, so we showed that $X \times Y$ satisfies this property, but why do we call this a "universal property"?

Proposition 12.4. *Suppose W is a space with continuous maps $q_X : W \longrightarrow X$ and $q_Y : W \longrightarrow Y$ also satisfying the property of the product. Then W is homeomorphic to $X \times Y$.*

Proof. The universal property for $X \times Y$ gives us a map $f : W \longrightarrow X \times Y$.

$$\begin{tikzcd}
& & X \\
W \arrow[r, dashed, "\exists! f"] \arrow[urr, bend left, "q_X"] \arrow[drr, bend right, "q_Y"'] & X \times Y \arrow[ur, "p_X"'] \arrow[dr, "p_Y"] & \\
& & Y
\end{tikzcd}$$

But W also has a universal property, so we get a map $\varphi : X \times Y \longrightarrow W$ as well.

$$\begin{tikzcd}
& & X \\
X \times Y \arrow[r, dashed, "\exists! \varphi"] \arrow[urr, bend left, "p_X"] \arrow[drr, bend right, "p_Y"'] & W \arrow[ur, "q_X"'] \arrow[dr, "q_Y"] & \\
& & Y
\end{tikzcd}$$

Now make Pacman eat Pacman!

$$
\begin{tikzcd}
 & & & X \\
W \arrow[r, dotted, "f"] \arrow[urrr, bend left, "q_X"] \arrow[drrr, bend right, "q_Y"'] & X \times Y \arrow[r, dotted, "\varphi"] \arrow[urr, dotted, "p_X"] \arrow[drr, dotted, "p_Y"'] & W \arrow[ur, "q_X"'] \arrow[dr, "q_Y"] & \\
 & & & Y
\end{tikzcd}
$$

We have a big diagram, but if we ignore all dotted lines, there is an obvious horizontal map $W \longrightarrow W$ to fill in the diagram, namely the id_W. Since the universal property guarantees that there is a **unique** way to fill it in, we find that $\varphi \circ f = \mathrm{id}_W$. Reversing the pacmen gives the other equality $f \circ \varphi = \mathrm{id}_{X \times Y}$. In other words, f is a homeomorphism, and $\varphi = f^{-1}$. ∎

This argument may seem strange the first time you see it, but it is a typical argument that applies any time you define an object via a universal property. The argument shows that any two objects satisfying the universal property must be "the same".

Proposition 12.5. *Let $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ be continuous. Then the product map $f \times f' : X \times X' \longrightarrow Y \times Y'$ is also continuous.*

Proof. This follows very easily from the universal property. If we want to map continuously to $Y \times Y'$, it suffices to specify continuous maps to Y and Y'. The continuous map $X \times X' \longrightarrow Y$ is the composition

$$X \times X' \xrightarrow{p_X} X \xrightarrow{f} Y,$$

and the other needed map is the composition

$$X \times X' \xrightarrow{p_{X'}} X' \xrightarrow{f'} Y'.$$

∎

Ok, so we understand $X \times Y$ as a topological space. What about a product of more than two spaces? Well, if we have a finite collection $X_1, \ldots, X_n$ of spaces, the product topology on $X_1 \times \cdots \times X_n$ has basis given by the $U_1 \times \cdots \times U_n$, or equivalently, prebasis given by the $p_j^{-1}(U_j)$. Note that this is equivalent because the basis element $U_1 \times \cdots \times U_n$, is a finite intersection of the prebasis elements $p_j^{-1}(U_j)$.

But what about the product of an *arbitrary* number of spaces? Here, the property we want is that whenever we have a space Z and maps $f_j : Z \longrightarrow X_j$ for all i, then there should be a unique continuous map $f : Z \longrightarrow \prod_{j \in J} X_j$ such that $p_j \circ f = f_j$.

Just as for finite products, we want the projection maps $p_j : \prod_{j \in J} \longrightarrow X_j$ to be continuous. This forces each $p_j^{-1}(U_j)$ to be continuous, and we can again choose these for a prebasis. We thus get a basis consisting of finite intersections $p_{j_1}^{-1}(U_{j1}) \cap \cdots \cap p_{j_k}^{-1}(U_{j_k})$.

Definition 12.6. Given spaces X_j, one for each $j \in J$, the **product topology** on $\prod_{j \in J} X_j$ has basis consisting of the $p_{j_1}^{-1}(U_{j1}) \cap \cdots \cap p_{j_k}^{-1}(U_{j_k})$.