Mon, Sept. 25

Last time, we introduced the *product topology* on $\prod_{j\in J} X_\alpha$, which had basis

$$\mathcal{B}_{\text{prod}} = \left\{ \prod_j U_j \mid U_j \subseteq X_j \text{ is open, and only finitely many } U_j \text{ are } \textbf{proper} \text{ subsets} \right\}.$$

Proposition 12.7. *The product topology on $\prod_{j\in J} X_j$, as defined above, satisfies the following universal property: given any space Z and continuous maps $f_j : Z \longrightarrow X_j$ for all $j \in J$, there is a unique continuous $f : Z \longrightarrow \prod_{j\in J} X_j$ such that $p_j \circ f = f_j$ for all $j \in J$.*

Proof. The same proof as that given in 12.3 works here. Given the maps f_j, we define f by $f(z)_j = f_j(z)$. Again, the equations $p_j \circ f = f_j$ force this choice on us. The only question is whether this makes f into a continuous map. Since the topology on $\prod_{j\in J} X_j$ is defined by the prebasis elements $p_j^{-1}(U_j)$, it suffices to show that each of these pulls back to an open set. But

$$f^{-1}(p_j^{-1}(U_j)) = (p_j \circ f)^{-1}(U_j) = f_j^{-1}(U_j),$$

which is open since f_j is continuous. ∎

12.1. **Box Topology.** We have defined the *product topology* on $\prod_{j\in J} X_j$. But there is another obvious guess, coming from the answer for finite products. We can think about the basis consisting of products $\prod_j U_j$. This is no longer equivalent to the product topology!

Definition 12.8. Suppose given a collection of spaces X_j. The **box topology** on $\prod_{j\in J} X_j$ is generated by the basis $\left\{ \prod_{j\in J} U_j \right\}$.

As discussed above, the box topology has more open sets; in other words, the box topology is finer than the product topology. To see that the box topology does not have the universal property we want, consider the following example: let $\Delta : \mathbb{R} \longrightarrow \prod_{n\in\mathbb{N}} \mathbb{R}$ be the diagonal map, all of whose component maps are simply the identity. For each n, let $I_n = (\frac{-1}{n}, \frac{1}{n})$. In the box topology, the subset $I = \prod_n I_n \subseteq \prod_n \mathbb{R}$ is an open set, but

$$\Delta^{-1}(I) = \bigcap_n \text{id}^{-1}(I_n) = \bigcap_n I_n = \{0\}$$

is not open. So the diagonal map is not continuous in the box topology!

Since we are now considering arbitrary products, it may be useful to stop and clarify what we mean. For instance, we might want to consider a countable infinite product $\mathbb{R} \times \mathbb{R} \times \dots$.

Let X_j, for $j \in J$, be sets. The cartesian product $\prod_{j\in J} X_j$ is the collection of tuples (x_j), where $x_j \in X_j$. This means that for each $j \in J$, we want an element $x_j \in X_j$. In other words, we should have a function

$$x_{(-)} : J \longrightarrow X = \bigcup_j X_j$$

with the condition that this function satisfies $x_j \in X_j$. With this language, the "projection" $\prod_{j \in J} X_j \longrightarrow X_j$ is simply the restriction along $\{j\} \hookrightarrow J$.

In the case that all X_j are the same set X, then $\prod_{j\in J} X_j$ is simply the set of functions $J \longrightarrow X$. So, the countably infinite product of $\mathbb{R}$ with itself is synonymous with the collection of sequences in $\mathbb{R}$.

Example 12.9. We mentioned above that the set of sequences in $\mathbb{R}$ is the infinite product $\prod_n \mathbb{R}$. What does a neighborhood of a sequence (x_n) look like in the product topology? We are only allowed to constrain finitely many coordinates, so a neighborhood consists of all sequences that are near to (x_n) in some fixed, finitely many coordinates.

Wed, Sept. 27

Proposition 12.10. *Let $A_j \subseteq X_j$ for all $j \in J$. Then*

$$\prod_j \overline{A_j} = \overline{\prod_j A_j}$$

in both the product and box topologies.

Proof. As usual, we have two subsets of $\prod_j X_j$ we want to show are the same, so we establish that each is a subset of the other. The following proof works in both topologies under consideration.

($\subseteq$) Let $(x_j) \in \prod \overline{A_j}$. We use the neighborhood criterion of the closure to show that $(x_j) \in \overline{\prod_j A_j}$. Thus let $U = \prod_j U_j$ be a basic open neighborhood of (x_j). Then for each j, U_j is a neighborhood of x_j. Since $x_j \in \overline{A_j}$, it follows that U_j must meet A_j in some point, say y_j. It then follows that $(y_j) \in U \cap \prod_j A_j$. By the neighborhood criterion, it follows that $(x_j) \in \overline{\prod_j A_j}$.

($\supseteq$) For the other direction, we simply use that the projection is continuous:

$$p_j\left(\overline{\prod_j A_j}\right) \subseteq \overline{p_j\left(\prod_j A_j\right)} = \overline{A_j}.$$

This shows that

$$\overline{\prod_j A_j} \subseteq \prod_j \overline{A_j}.$$

■

Note that this implies that an (arbitrary) product of closed sets is closed, using either the product or box topologies. In particular, I^2 is closed in $\mathbb{R}^2$ and T^2 is closed in $\mathbb{R}^4$.

Proposition 12.11. *Suppose X_j is Hausdorff for each $j \in J$. Then so is $\prod_j X_j$ in both product and box topologies.*

Proof. Let $(x_j) \neq (x'_j) \in \prod_j X_j$. Then $x_\ell \neq x'_\ell$ for some particular ℓ. Since X_ℓ is Hausdorff, we can find disjoint neighborhoods U and U' of x_ℓ and x'_ℓ in X_ℓ. Then $p_\ell^{-1}(U)$ and $p_\ell^{-1}(U')$ are disjoint neighborhoods of (x_j) and (x'_j) in the product topology, so $\prod_j X_j$ is Hausdorff in the product topology.

For the box topology, we can either say that the above works just as well for the box topology, or we can say that since the box topology is a refinement of the product topology and the product topology is Hausdorff, it follows that the box topology must also be Hausdorff. ■

The converse is true as well, assuming that each X_j is nonempty. To see this, we use the fact that a subspace of a Hausdorff space is Hausdorff. How do we view X_ℓ as a subspace of $\prod_j X_j$? We can think about an axis inclusion. Thus pick $y_j \in X_j$ for $j \neq \ell$. We define

$$a_\ell : X_\ell \longrightarrow \prod_j X_j$$

by

$$a_\ell(x)_j = \begin{cases} x & j = \ell \\ y_j & j \neq \ell. \end{cases}$$

Note that, by the universal property of the product, in order to check that a_ℓ is continuous, it suffices to check that each coordinate map is continuous. But the coordinate maps are the identity and a lot of constant maps, all of which are certainly continuous. The map a_ℓ is certainly injective (assuming all X_j are nonempty!), and it is an example of an embedding.

Definition 12.12. A map $f : X \longrightarrow Y$ is said to be an **embedding** if it is a homeomorphism onto its image $f(X)$, equipped with the subspace topology.

We already discussed injectivity and continuity of the axis inclusion a_ℓ, so it only remains to show this is open, as a map to $a_\ell(X_\ell)$. Let $U \subseteq X_\ell$ be open. Then

$$a_\ell(U) = p_\ell^{-1}(U) \cap a_\ell(X_\ell),$$

so $a_\ell(U)$ is open in the subspace topology on $a_\ell(X_\ell)$.

We will often do the above sort of exercise: if we introduce a new property or construction, we will ask how well this interacts with other constructions/properties.

Here is another example of an embedding.

Example 12.13. Let $f : X \longrightarrow Y$ be continuous and define the graph of f to be

$$\Gamma(f) = \{(x, y) \mid y = f(x)\} \subseteq X \times Y.$$

The function

$$\gamma : X \longrightarrow X \times Y, \qquad \gamma(x) = (x, f(x))$$

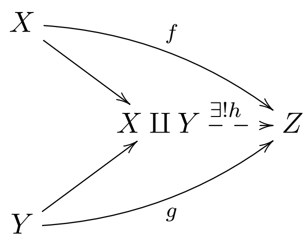
is an embedding with image $\Gamma(f)$.

Let us verify that this is indeed an embedding. Injectivity is easy (this follows from the fact that one of the coordinate maps is injective), and continuity comes from the universal property for the product $X \times Y$ since id_X and f are both continuous. Note that $(p_X)_{|\Gamma(f)}$, which is continuous since it is the restriction of the continuous projection p_X, provides an inverse to γ.

Fri, Sept. 29

13. Coproduct

What happens if we turn all of the arrows around in the defining property of a product? We might call such a thing a "coproduct". To be precise we would want a space that is universal among spaces equipped with maps *from* X and Y. In other words, given a space Z and maps $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$, we would want a unique map from the coproduct to Z, making the following diagram commute.



The glueing lemma gave us exactly such a description, in the case that our domain space X was made up of *disjoint* open subsets A and B. In general, the answer here is given by the **disjoint union**.

Recall that, as a set, the disjoint union of sets X and Y is the subset

$$X \amalg Y \subseteq (X \cup Y) \times \{1, 2\},$$

where $X \amalg Y = (X \times \{1\}) \cup (Y \times \{2\})$. More generally, given sets X_j for $j \in J$, their disjoint union $\coprod_j X_j$ is the subset

$$\coprod_j X_j \subseteq \left(\bigcup_j X_j \right) \times J$$

given by

$$\coprod_j X_j = \bigcup_j (X_j \times \{j\}) \,.$$

There are natural inclusions $\iota_X : X \longrightarrow X \amalg Y$ or more generally $\iota_{X_j} : X_j \hookrightarrow \coprod_j X_j$. We topologize the coproduct by giving it the finest topology such that all ι_{X_j} are continuous. In other words, a subset $U \subseteq \coprod_j X_j$ is open if and only if $\iota_j^{-1}(U) \subseteq X_j$ is open for all j.

Note that in the case of a coproduct of two spaces, the subspace topology on $X \subseteq X \amalg Y$ agrees with the original topology on X. Furthermore, both X and Y are open in $X \amalg Y$, so the universal property for the coproduct is precisely the glueing lemma.

On Friday, we introduced the idea of a coproduct, which is dual to the product. In the case of a space X which happens to be the union of two open, disjoint, subspaces A and B, then the glueing lemma told us that X satisfies the correct property to be the coproduct $X = A \amalg B$.

For a more general coproduct $\coprod_j X_j$, we declared $U \subseteq \coprod_j X_j$ to be open if and only if $\iota_j^{-1}(U)$ is open for all j. Let's verify that this satisfies the universal property.

Thus let $f_j : X_j \longrightarrow Z$ be continuous for all $j \in J$. It is clear that, set-theoretically, the various images $\iota_j(X_j)$ inside the coproduct are disjoint and that their union is the entire coproduct. So to define a function on the coproduct, it suffices to define a function on each $\iota_j(X_j)$. But each ι_j is injective, in other words a bijection onto its image, so defining $f_{|\iota_j(X_j)}$ is equivalent to defining

$f_{|\iota_j(X_j)} \circ \iota_j$. But the latter, according to the universal property, is supposed to be f_j. So the upshot of all of this is that there is no choice in how we define the function f. As usual, we only need verify that this function f is continuous.

Let $V \subseteq Z$ be open. We wish to know that $f^{-1}(V)$ is open in $\coprod_j X_j$. But according to the topology on the coproduct, this amounts to showing that each $\iota_j^{-1} f^{-1}(V)$ is open. But this is $(f \circ \iota_j)^{-1}(V) = f_j^{-1}(V)$, which is open by the assumption that each f_j is continuous.

Example 13.1.

(1) Consider $X = [0, 1]$ and $Y = [2, 3]$. Then in this case $X \amalg Y$ is homeomorphic to the subspace $X \cup Y$ of $\mathbb{R}$. The same is true of these two intervals are changed to be open or half-open.

(2) Consider $X = (0, 1)$ and $Y = \{1\}$. Then $X \amalg Y$ is **not** homeomorphic to $(0, 1) \cup \{1\} = (0, 1]$. The singleton $\{1\}$ is open in $X \amalg Y$ but not in $(0, 1]$. Instead, $X \amalg Y$ is homeomorphic to $(0, 1) \cup \{2\}$.

(3) Similarly $(0, 1) \amalg [1, 2]$ is homeomorphic to $(0, 1) \cup [2, 3]$ but not to $(0, 1) \cup [1, 2] = (0, 2]$.

(4) In yet another similar example, $(0, 2) \amalg (1, 3)$ is homeomorphic to $(0, 1) \cup (2, 3)$ but not to $(0, 2) \cup (1, 3) = (0, 3)$.

Proposition 13.2. *Let X_i be spaces, for $i \in I$. Then $\coprod_i X_i$ is Hausdorff if and only if all X_i are Hausdorff.*