Mon, Oct. 2

Proof. This is even easier than for products. First, X_i always embeds as a subspace of the coproduct, so it follows that X_i is Hausdorff if the coproduct is as well. On the other hand, suppose all X_i are Hausdorff and suppose that $x \neq y$ are points of $\coprod_i X_i$. Either x and y come from different X_i's, in which case the X_i's themselves serve as the disjoint neighborhoods. The alternative is that x and y live in the same Hausdorff X_i, but then we can find disjoint neighborhoods in X_i. ■

14. Quotients

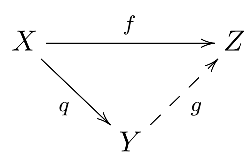
The next important construction is that of a quotient, or identification space.

The general setup is that we have a surjective map $q : X \longrightarrow Y$, which we view as making an identification of points in X. More precisely, suppose that we have an equivalence relation $\sim$ on X. We can consider the set $X/\sim$ of equivalence classes in X. There is a natural surjective map $q : X \longrightarrow X/\sim$ which takes $x \in X$ to its equivalence class.

And in fact every surjective map is of this form. Suppose that $q : X \longrightarrow Y$ is surjective. We define a relation on X by saying that $x \sim x'$ if and only if $q(x) = q(x')$. Then the function $X/\sim \longrightarrow Y$ sending the class of x to $q(x)$ is a bijection.

We want to mimic the above situation in topology, but to understand what this should mean, we first look at the universal property of the quotient for sets. This says: if $f : X \longrightarrow Z$ is a function that is constant on the equivalence classes in X, then there is a (unique) factorization $g : X/\sim \longrightarrow Z$ with $g \circ q = f$.

We want to have a similar setup in topology. Said in the equivalence relation framework, given a space X and a relation $\sim$ on X, we want a continuous map $q : X \longrightarrow Y$ such that given any space Z with a continuous map $f : X \longrightarrow Z$ which is constant on equivalence classes, there is a unique continuous map $g : Y \longrightarrow Z$ such that $g \circ q = f$.



By considering the cases in which Z is a set with the trivial topology, so that maps to Z are automatically continuous, we can see that on the level of sets $q : X \longrightarrow Y$ must be $X \longrightarrow X/\sim$. It remains only to specify the topology on $Y = X/\sim$.

We want the topological quotient to be the universal example of a continuous map out of X which is constant on equivalence classes. Since we want to construct maps *out of* Y, this suggests we should include as many open sets as possible in Y. This leads to the following definition.

Definition 14.1. We say that a surjective map $q : X \longrightarrow Y$ is a **quotient map** if $V \subseteq Y$ is open if and only if $q^{-1}(V)$ is open in X.

One implication is the definition of continuity, but the other is given by our desire to include as many opens as we can.

Proposition 14.2. *(Universal property of the quotient) Let $q : X \longrightarrow Y$ be a quotient map. If Z is any space, and $f : X \longrightarrow Z$ is any continuous map that is constant on the fibers*[2] *of q, then there exists a unique continuous $g : Y \longrightarrow Z$ such that $g \circ q = f$.*

[2] A “fiber” is simply the preimage of a point.

Proof. It is clear how g must be defined: $g(y) = f(x)$ for any $x \in q^{-1}(y)$. It remains to show that g is continuous. Let $W \subseteq Z$ be open. We want $g^{-1}(W) \subseteq Y$ to be open as well. By the definition of a quotient map, $g^{-1}(W)$ is open if and only if $q^{-1}(g^{-1}(W)) = (g \circ q)^{-1}(W) = f^{-1}(W)$ is open, so we are done by continuity of f. ■

Example 14.3. Define $q : \mathbb{R} \longrightarrow \{-1, 0, 1\}$ by

$$q(x) = \begin{cases} 0 & x = 0 \\ \frac{|x|}{x} & x \neq 0. \end{cases}$$

What is the resulting topology on $\{-1, 0, 1\}$? The points -1 and 1 are open, and the only open set containing 0 is the whole space.

Note that this example shows that a quotient of a Hausdorff space need not be Hausdorff.

Proposition 14.4. *Let $q : X \longrightarrow Y$ be a continuous, surjective, open map. Then q is a quotient map. The same is true if q is closed instead of open.*

Proof. One implication is simply the definition of continuity. For the other, suppose that $V \subseteq Y$ is a subset such that $q^{-1}(V) \subseteq X$ is open. Then $q(q^{-1}(V))$ is open since q is open. Finally, we have $V = q(q^{-1}(V))$ since q is surjective. ■

The converse is not true, however, as the next example shows.

Example 14.5. Consider $q; \mathbb{R} \longrightarrow [0, \infty)$ given by

$$q(x) = \begin{cases} 0 & x \leq 0 \\ x & x \geq 0. \end{cases}$$

The quotient topology on $[0, \infty)$ is the same as the subspace topology it gets from $\mathbb{R}$. But this is not an open map, since the image of $(-2, -1)$ is $\{0\}$, which is not open.

14.1. **Saturated Open Sets.** We discussed last time the fact that a quotient map need not be open. Nevertheless, there is a class of open sets that are always carried to open sets.

Definition 14.6. Let $q : X \longrightarrow Y$ be a continuous surjection. We say a subset $A \subseteq X$ is **saturated** (with respect to q) if it is of the form $q^{-1}(V)$ for some subset $V \subseteq Y$.

Wed, Oct. 4

It follows that A is saturated if and only if $q^{-1}(q(A)) = A$. Recall that a **fiber** of a map $q : X \longrightarrow Y$ is the preimage of a single point. Then another description is that A is saturated if and only if it contains all fibers that it meets.

Proposition 14.7. *A continuous surjection $q : X \twoheadrightarrow Y$ is a quotient map if and only if it takes saturated open sets to saturated open sets.*

Proof. Exercise. ■

Example 14.8. (Collapsing a subspace) Let $A \subseteq X$ be a subspace. We define a relation on X as follows: $x \sim y$ if both are points in A or if neither is in A and $x = y$. Here, we have one equivalence class for the subset A, and every point outside of A is its own equivalence class. Standard notation for the set $X/\sim$ of equivalence classes under this relation is X/A. The universal property can be summed up as saying that any map on X which is constant on A factors through the quotient X/A.

For example, we considered last time the example $\mathbb{R}/(-\infty, 0] \cong [0, \infty)$.

Example 14.9. Consider $\partial I \subseteq I$. The exponential map $e : I \longrightarrow S^1$ is constant on ∂I, so we get an induced continuous map $\varphi : I/\partial I \longrightarrow S^1$, which is easily seen to be a bijection. In fact, it is a homeomorphism. Once we learn about compactness, it will be easy to see that this is a closed map.

We show instead that it is open. A basis for $I/\partial I$ is given by $q(a,b)$ with $0 < a < b < 1$ and by $q([0,a) \cup (b,1])$ with again $0 < a < b < 1$. Since both are taken to basis elements for the subspace topology on S^1, it follows that φ is a homeomorphism.

Example 14.10. Generalizing the previous example, for any closed ball $D^n \subseteq \mathbb{R}^{n+1}$, we can consider the quotient $D^n/\partial D^n$. Exercise: define a surjective continuous map

$$q : D^n \longrightarrow S^n$$

taking the origin to the south pole and the boundary to the north pole. This then defines a continuous bijection $D^n/\partial D^n \longrightarrow S^n$, and we will see later in the course that this is automatically a homeomorphism.

Fri, Oct. 6

Exam day