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(**Examples continued . . .**)

Example 14.11. (Cylinder) On $I \times I$, we impose the relation $(0, y) \sim (1, y)$. The resulting quotient space is a cylinder, which can be identified with $S^1 \times I$.

Example 14.12. (Möbius band) On $I \times I$, we impose the relation $(0, y) \sim (1, 1-y)$. The resulting quotient space is the Möbius band M.

Example 14.13. (Torus) On $I \times I$, we impose the relation $(0, y) \sim (1, y)$ and also the relation $(x, 0) \sim (x, 1)$. The resulting quotient space is the torus $T^2 = S^1 \times S^1$. We recognize this as the product of two copies of example 14.8, but beware that in general a product of quotient maps need not be a quotient map.

Example 14.14. (Real projective space) On S^n we impose the equivalence relation $\mathbf{x} \sim -\mathbf{x}$. The resulting quotient space is known as n-dimensional real projective space and is denoted $\mathbb{RP}^n$.

Consider the case $n = 1$. We have the hemisphere inclusion $I \hookrightarrow S^1$ given by $x \mapsto e^{ix\pi}$. Then the composition $I \hookrightarrow S^1 \twoheadrightarrow \mathbb{RP}^1$ is a quotient map that simply identifies the boundary ∂I to a point. In other words, this is example 14.8 from above, and we conclude that $\mathbb{RP}^1 \cong S^1$. However, the higher-dimensional versions of these spaces are certainly not homeomorphic. We will return to this soon in Example 15.5.

Example 14.15. (Complex projective space) Consider S^{2n-1} as a subspace of $\mathbb{C}^n$. We then have the coordinate-wise multiplication by elements of $S^1 \cong U(1)$ on $\mathbb{C}^n$. This multiplication restricts to a multiplication on the subspace S^{2n-1}, and we impose an equivalence relation $(z_1, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n)$ for all $\lambda \in S^1$. The resulting quotient space is the complex projective space $\mathbb{CP}^n$.

15. Topological Groups

A number of the examples above have secretly been examples of a more general construction, namely the quotient under the action of a group.

Definition 15.1. A **topological group** is a based space (G, e) with a continuous multiplication $m : G \times G \longrightarrow G$ and inverse $i : G \longrightarrow G$ satisfying all of the usual axioms for a group.

Remark 15.2. Munkres requires all topological groups to satisfy the condition that points are closed. We will not make this restriction, though the examples we will consider will all satisfy this.

Example 15.3.

(1) Any group G can be considered as a topological group equipped with the discrete topology. For instance, we have the cyclic groups $\mathbb{Z}$ and $C_n = \mathbb{Z}/n\mathbb{Z}$.
(2) The real line $\mathbb{R}$ is a group under addition, This is a topological group because addition and multiplication by -1 are both continuous. Note that here $\mathbb{Z}$ is at the same time both a subspace and a subgroup. It is thus a topological subgroup.
(3) If we remove zero, we get the multiplicative group $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ of real numbers.

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(4) Inside $\mathbb{R}^\times$, we have the subgroup $\{1, -1\}$ of order two.
(5) $\mathbb{R}^n$ is also a topological group under addition. In the case $n = 2$, we often think of this as $\mathbb{C}$.
(6) Again removing zero, we get the multiplicative group $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ of complex numbers.
(7) Inside $\mathbb{C}^\times$ we have the subgroup of complex numbers of norm 1, aka the circle group $S^1 \cong U(1) = SO(2)$.

(8) This last example suggests that matrix groups in general are good candidates. For instance, we have the topological group $Gl_n(\mathbb{R})$. This is a subspace of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. The determinant mapping $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is polynomial in the coefficients and therefore continuous. The general linear group is the complement of $\det^{-1}(0)$. It follows that $Gl_n(\mathbb{R})$ is an open subspace of $\mathbb{R}^{n^2}$.

(9) Inside $Gl_n(\mathbb{R})$, we have the closed subgroups $Sl_n(\mathbb{R})$, $O(n)$, $SO(n)$.

Let G be a topological group and fix some $h \in G$. Define $L_h : G \longrightarrow G$ by $L_h(g) = hg$. This is left multiplication by h. The definition of topological group implies that this is continuous, as L_h is the composition

$$G \xrightarrow{(h,\mathrm{id})} G \times G \xrightarrow{m} G.$$

Moreover, $L_{h^{-1}}$ is clearly inverse to L_h and continuous by the same argument, so we conclude that each L_h is a homeomorphism. Since $L_h(e) = h$, we conclude that neighborhoods around h look like neighborhoods around e. Since h was arbitrary, we conclude that neighborhoods around one point look like neighborhoods around any other point. This implies that a space like the unoin of the coordinate axes in $\mathbb{R}^2$ cannot be given the structure of topological group, as neighborhoods around the origin do not resemble neighborhoods around other points.

The main reason for studying topological groups is to consider their *actions* on spaces.

Definition 15.4. Let G be a topological group and X a space. A **left action** of G on X is a map $a : G \times X \longrightarrow X$ which is associative and unital. This means that $a(g, a(h, x)) = a(gh, x)$ and $a(e, x) = x$. Diagrammatically, this is encoded as the following commutative diagrams

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{\mathrm{id}\times a} & G \times X \\ {\scriptstyle m\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle a} \\ G \times X & \xrightarrow[a]{} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{e,\mathrm{id}} & G \times X \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle a} \\ & & X. \end{array}$$

It is common to write $g \cdot x$ or simply gx rather than $a(g, x)$.

There is a similar notion of right action of G on X, given by a map $X \times G \longrightarrow X$ satisfying the appropriate properties.

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Given an action of G on a space X, we define a relation on X by $x \sim y$ if $y = g \cdot x$ for some g. The equivalence classes are known as **orbits** of G in X, and the quotient of X by this relation is typically written as X/G. Really, the notation X/G should be reserved for the quotient by a *right action* of G on X, and the quotient by a left action should be $G\backslash X$.

Example 15.5. (1) For any G, left multiplication gives an action of G on itself! This is a transitive action, meaning that there is only one orbit, and the quotient $G\backslash G$ is just a point.

Note that we saw above that, for each $h \in G$, the map $L_h : G \longrightarrow G$ is a homeomorphism. This generalizes to any action. For each $g \in G$, the map $a(g, -) : X \longrightarrow X$ is a homeomorphism.

(2) For any (topological) subgroup $H \leq G$, left multiplication by elements of H gives a left action of H on G. Note that an orbit here is precisely a right coset Hg. The quotient is $H\backslash G$, the set of right cosets of H in G.

(3) Consider the subgroup $\mathbb{Z} \leq \mathbb{R}$. Since $\mathbb{R}$ is abelian, we don't need to worry about about left vs. right actions or left vs. right cosets. We then have the quotient $\mathbb{R}/\mathbb{Z}$, which is again a topological group (again, $\mathbb{R}$ is abelian, so $\mathbb{Z}$ is normal).

What is this group? Once again, consider the exponential map $\exp : \mathbb{R} \longrightarrow S^1$ given by $\exp(x) = e^{2\pi i x}$. This is surjective, and it is a homomorphism since

$$\exp(x+y) = \exp(x)\exp(y).$$

The First Isomorphism Theorem in group theory tells us that $S^1 \cong \mathbb{R}/\ker(\exp)$, at least as a group. The kernel is precisely $\mathbb{Z} \leq \mathbb{R}$, and it follows that $S^1 \cong \mathbb{R}/\mathbb{Z}$ as a group. To see that this is also a homeomorphism, we need to know that $\exp : \mathbb{R} \longrightarrow S^1$ is a quotient map, but this follows from our earlier verification that $I \longrightarrow S^1$ is a quotient. Another way to think about this is that the universal property of the quotient gives us continuous maps $I/\partial I \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow I/\partial I$ which are inverse to each other.

(4) Similarly, we can think of $\mathbb{Z}^n$ acting on $\mathbb{R}^n$, and the quotient is $\mathbb{R}^n/\mathbb{Z}^n \cong (S^1)^n = T^n$.

(5) The group $Gl(n)$ acts on $\mathbb{R}^n$ (just multiply a matrix with a vector), but this is not terribly interesting, as there are only two orbits: the origin is a closed orbit, and the complement is an open orbit. Thus the quotient space consists of a closed point and an open point.

(6) More interesting is the action of the subgroup $O(n)$ on $\mathbb{R}^n$. Using the fact that orthogonal matrices preserve norms, it is not difficult to see that the orbits are precisely the spheres around the origin. We claim that the quotient is the space $[0,\infty)$ (thought of as a subspace of $\mathbb{R}$).

To see this, consider the continuous surjection $|-| : \mathbb{R}^n \longrightarrow [0,\infty)$. By considering how this acts on open balls, you can show that this is an open map and therefore a quotient. But the fibers of this map are precisely the spheres, so it follows that this is the quotient induced by the above action of $O(n)$.

(7) Let $\mathbb{R}^\times$ act on $\mathbb{R}^n$ via scalar multiplication. This action preserves lines, and within each line there are two orbits, one of which is the origin. Note that the only saturated open set containing 0 is $\mathbb{R}^n$, so the only neighborhood of 0 in the quotient is the entire space.

(8) Switching from n to $n+1$ for convenience, we can remove that troublesome 0 and let $\mathbb{R}^\times$ act on $X_{n+1} = \mathbb{R}^{n+1} \setminus \{0\}$. Here the orbits are precisely the lines (with origin removed). The quotient is $\mathbb{RP}^n$.

To see this, recall that we defined $\mathbb{RP}^n$ as the quotient of S^n by the relation $\mathbf{x} \sim -\mathbf{x}$. This is precisely the relation that arises from the action of the subgroup $C_2 = \{1,-1\} \leq \mathbb{R}^\times$ on $S^n \subseteq \mathbb{R}^{n+1}$.

Now notice that the map $\mathbb{R}^{n+1} \setminus \{0\} \longrightarrow S^n \times \mathbb{R}_{>0}$ given by $\mathbf{x} \mapsto \left(\frac{\mathbf{x}}{\|\mathbf{x}\|}, \|\mathbf{x}\|\right)$ is a homeomorphism. Next, note that we have an isomorphism $\mathbb{R}^\times \cong C_2 \times \mathbb{R}^\times_{>0}$. Thus the quotient $(\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^\times$ can be viewed as the two step quotient $\left((S^{n-1} \times \mathbb{R}_{>0})/\mathbb{R}^\times_{>0})\right)/C_2$. But $(\mathbb{R}^{n-1} \times \mathbb{R}_{>0})/\mathbb{R}^\times_{>0} \cong S^{n-1}$, so we are done.

We can think of $\mathbb{RP}^n$ in yet another way. Consider the following diagram:

$$\begin{array}{ccccc} D^n & \longrightarrow & S^n & \longrightarrow & \mathbb{R}^{n+1} \setminus \{0\} \\ \downarrow & & \downarrow & & \downarrow \\ D^n/\sim & \longrightarrow & S^n/C_2 & \longrightarrow & (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^\times \end{array}$$

The map $D^n \longrightarrow S^n$ is the inclusion of a hemisphere. The relation on D^n is the relation $\mathbf{x} \sim -\mathbf{x}$, but only allowed *on the boundary* ∂D^n. All maps on the bottom are continuous bijections, and again we will see later that they are necessarily homeomorphisms.

Note that the relation we imposed on D^n does *not* come from an action of C_2 on D^n. Let us write $C_2 = \langle \sigma \rangle$. We can try defining

$$\sigma \cdot \mathbf{x} = \begin{cases} \mathbf{x} & \mathbf{x} \in \mathrm{Int}(D^n) \\ -\mathbf{x} & \mathbf{x} \in \partial(D^n), \end{cases}$$

where here the interior and boundary are taken in S^n. But this is not continuous, as the convergent sequence

$$\left(\sqrt{1-\frac{1}{n}}, 0, \ldots, 0, \sqrt{\frac{1}{n}}\right) \to (1, 0, \ldots, 0)$$

is taken by σ to a convergent sequence, but the new limit is not $\sigma(1, 0, \ldots, 0) = (-1, 0, \ldots, 0)$.