$\boxed{\textbf{Mon, Oct. 16}}$

Last time, we discussed real projective space as a quotient $\mathbb{RP}^n \cong (\mathbb{R}^{n+1} - \{0\})/\mathbb{R}^\times$. We have a similar story for $\mathbb{CP}^n$.

Example 15.6. There is an action of $\mathbb{C}^\times$ on $\mathbb{C}^{n+1} \setminus \{0\}$, and the orbits are the punctured complex lines. We claim that the quotient is $\mathbb{CP}^n$.

We defined $\mathbb{CP}^n$ as a quotient of an S^1-action on S^{2n+1}. We also have a homeomorphism $\mathbb{C}^{n+1} \setminus \{0\} \cong S^{2n+1} \times \mathbb{R}_{>0}$ and an isomorphism $\mathbb{C}^\times \cong S^1 \times \mathbb{R}^\times_{>0}$. We can then describe $\mathbb{CP}^n$ as the two-step quotient

$$\big(\mathbb{C}^{n+1} \setminus \{0\}\big)/\mathbb{C}^\times \cong \Big((S^{2n+1} \times \mathbb{R}_{>0})/\mathbb{R}^\times_{>0}\Big)/S^1 \cong S^{2n+1}/S^1 = \mathbb{CP}^n.$$

We have been studying actions of topological groups on spaces, and the resulting quotient spaces X/G. But there is another way to think about this material. Suppose you have a set Y that you would like to topologize. One way to create a topology on Y is as follows. Pick a point $y_0 \in Y$. If there is a transitive action of some topological group G on Y, then the orbit-stabilizer theorem asserts that Y can be identified with G/H, where $H \le G$ is the stabilizer subgroup consisting of all $h \in G$ such that $h \cdot y_0 = y_0$. But G/H is a topological space, so we define the topology on Y to be the one coming from the bijection $Y \cong G/H$.

Example 15.7. (Grassmannian) We saw that the projective spaces can be identified with the set of lines in $\mathbb{R}^n$ or $\mathbb{C}^n$, suitably topologized. We can similarly consider the set of k-dimensional linear subspaces in $\mathbb{R}^n$ (or $\mathbb{C}^n$). It is not clear how to topologize this set.

However, there is an action of $O(n)$ on the set of k-planes in $\mathbb{R}^n$. Really, this comes from an action of the larger group $Gl_n(\mathbb{R})$, but the $O(n)$-action turns out to be more convenient. Namely, if $A \in O(n)$ is an orthogonal matrix and $V \subseteq \mathbb{R}^n$ is a k-dimensional subspace, then $A(V) \subseteq \mathbb{R}^n$ is another k-dimensional subspace. Furthermore, this action is transitive. To see this, it suffices to show that given any subspace V, there is a matrix taking the standard subspace $E_k = \mathrm{Span}\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$ to V. Thus suppose $V = \mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a k-dimensional subspace with given orthonormal basis. This can be completed to an orthonormal basis of $\mathbb{R}^n$. Then if A is the orthogonal matrix with columns the $\mathbf{v}_i$, A takes the standard subspace E_k to V.

The stabilizer of E_k is the subgroup of orthogonal matrices that take E_k to E_k. Such matrices are block matrices, with an orthogonal $k \times k$ matrix in the upper left and an orthogonal $(n-k)\times(n-k)$ matrix in the lower right. In other words, the stabilizer subgroup is $O(k) \times O(n-k)$. It follows that the set of k-planes in $\mathbb{R}^n$ can be identified with the quotient

$$\mathrm{Gr}_{k,n}(\mathbb{R}) = O(n)/\Big(O(k) \times O(n-k)\Big).$$

Note that, from this identification, we can see that $\mathrm{Gr}_{k,n} \cong \mathrm{Gr}_{n-k,n}$. The map takes a k-plane in $\mathbb{R}^n$ to the orthogonal complement, which is an $n-k$-plane in $\mathbb{R}^n$. The corresponding map

$$O(n)/\Big(O(k) \times O(n-k)\Big) \longrightarrow O(n)/\Big(O(n-k) \times O(k)\Big)$$

is induced by a map $O(n) \longrightarrow O(n)$. This map on $O(n)$ is conjugation by a shuffle permutation that permutes k things past $n-k$ things.

There is an identical story for the complex Grasmannians, where $O(n)$ is replaced by $U(n)$.

Wed, Oct. 18

Part 4. Properties

16. Connectedness

What we have done so far corresponds roughly to Chapters 2 & 3 of Lee. Now we turn to Chapter 4.

The first idea is connectedness. Essentially, we want to say that a space cannot be decomposed into two disjoint pieces.

Definition 16.1. A **disconnection** (or separation) of a space X is a pair of disjoint, nonempty open subsets $U, V \subseteq X$ with $X = U \cup V$. We say that X is **connected** if it has no disconnection.

Example 16.2.

1. If X is a discrete space (with at least two points), then any pair of disjoint nonempty subsets gives a disconnection of X.
2. Let X be the subspace $(0,1) \cup (2,3)$ of $\mathbb{R}$. Then X is disconnected.
3. More generally, if $X \cong A \coprod B$ for nonempty spaces A and B, then X is disconnected.
4. Another example of a disconnected subspace of $\mathbb{R}$ is the subspace $\mathbb{Q}$. A disconnection of $\mathbb{Q}$ is given by $(-\infty, \pi) \cap \mathbb{Q}$ and $(\pi, \infty) \cap \mathbb{Q}$.
5. Any set with the trivial topology is connected, since there is only one nonempty open set.
6. Of the 29 topologies on $X = \{1,2,3\}$, 19 are connected, and the other 10 are disconnected. For example, the topology $\{\emptyset, \{1\}, X\}$ is connected, but $\{\emptyset, \{1\}, \{2,3\}, X\}$ is not.
7. If X is a space with the generic point (or included point) topology, in which the nonempty open sets are precisely the ones containing a special point x_0, then X is connected.
8. If X is a space with the excluded point topology, in which the open proper subsets are the ones missing a special point x_0, then X is connected.
9. The lower limit topology $\mathbb{R}_{\ell\ell}$ is disconnected, as the basis elements $[a,b)$ are both open and closed (clopen!), which means that their complements are open.

Proposition 16.3. *Let X be a space. The following are equivalent:*

1. *X is disconnected*
2. *$X \cong A \coprod B$ for nonempty spaces A and B*
3. *There exists a nonempty, clopen, proper subset $U \subseteq X$*
4. *There exists a continuous surjection $X \twoheadrightarrow \{0,1\}$, where $\{0,1\}$ has the discrete topology.*

Now let's look at an interesting example of a connected space.

Proposition 16.4. *The only (nonempty) connected subspaces of $\mathbb{R}$ are intervals (including singletons).*

Proof. Note that, by an interval, we mean simply a convex subset of $\mathbb{R}$. Any connected subset must be an interval since if A is connected and $a < b < c$ with $a, c \in A$, then either $b \in A$ or $(-\infty, b) \cap A$ and $(b, \infty) \cap A$ give a separation of A.

So it remains to show that intervals are connected. Singletons are connected, as there is only one nonempty subset. Thus let $I \subseteq \mathbb{R}$ be an interval with at least two points, and let $U \subseteq I$ be nonempty and clopen (in the subspace topology on I). We wish to show that $U = I$. Let $a \in U$. We will show that $U \cap [a, \infty) = I \cap [a, \infty)$. In other words, we wish to show that if $b > a$ and $b \in I$, then $b \in U$. A similar argument will show that $U \cap (-\infty, a] = I \cap (-\infty, a]$.

Consider the set

$$R_a = \{b \in I \mid [a,b] \subseteq U\}.$$

Note that $a \in R_a$, so that R_a is nonempty. If R_a is not bounded above, then $[a, \infty) \subseteq U \subseteq I$, and we have our conclusion.

Otherwise, the set R_a has a supremum $s = \sup R_a$ in $\mathbb{R}$. Note that if $s \notin I$, then since I is an interval, no real number larger y than s can be in I, since otherwise the entire interval $[a, y]$, which contains s, would be contained in I. Then

$$[a, s) \ \subseteq \ U \cap [a, \infty) \ \subseteq \ I \cap [a, \infty) = [a, s).$$

It follows that $U \cap [a, \infty) = I \cap [a, \infty) = [a, s)$.

The final case to consider is when $s \in I$. Since we can express s as a limit of a U-sequence and since U is closed in I, it follows that s must also lie in U. Since U is open, some ϵ-neighborhood of s (in I) lies in U. But no point in $(s, s + \epsilon/2)$ can lie in U (or I), since any such point would then also lie in R_a. Again, since I is an interval we have

$$U \cap [a, \infty) = [a, s] = I \cap [a, \infty).$$

■

Fri, Oct. 20

One of the most useful results about connected spaces is the following.

Proposition 16.5. *Let $f : X \longrightarrow Y$ be continuous. If X is connected, then so is $f(X) \subseteq Y$.*

Proof. Suppose that $U \subseteq f(X)$ is closed and open. Then $f^{-1}(U)$ must be closed and open, so it must be either $\emptyset$ or X. This forces $U = \emptyset$ or $U = f(X)$. ■

Since the exponential map $\exp : [0, 1] \longrightarrow S^1$ is a continuous surjection, it follows that S^1 is connected. More generally, we have

Proposition 16.6. *Let $q : X \longrightarrow Y$ be a quotient map with X connected. Then Y is connected.*

As another application, we have

Theorem 16.7 (Intermediate Value Theorem)**.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. Then f attains every intermediate value between $f(a)$ and $f(b)$.*

Proof. This follows from the fact that the image is connected and so must be an interval by Proposition 16.4. ■

Which of the other constructions we have seen preserve connectedness? All of them! (Well, except that subspaces of connected spaces need not be connected, as we have already seen.)

Proposition 16.8. *Let $A_i \subseteq X$ be connected for each i, and assume that $x_0 \in \bigcap_i A_i \neq \emptyset$. Then $\bigcup_i A_i$ is connected.*

Proof. Assume each A_i is connected, and let $U \subseteq \bigcup_i A_i$ be nonempty and clopen. Let $x \in U \subseteq \bigcup_i A_i$. Suppose $x \in A_{i_0}$. Then $U \cap A_{i_0}$ is nonempty and clopen in A_{i_0}, so $U \cap A_{i_0} = A_{i_0}$. In other words, $A_{i_0} \subseteq U$. Since $x_0 \in A_{i_0}$, it follows that $x_0 \in U$. But now for any other A_j, we have that $x_0 \in A_j \cap U$, so that $A_j \cap U$ is nonempty and clopen in A_j. It follows that $A_j \subseteq U$. ■

As an application, we get that products interact well with connectedness.

Proposition 16.9. *Assume $X_i \neq \emptyset$ for all $i \in \{1, \ldots, n\}$. Then $\prod_{i=1}^{n} X_i$ is connected if and only if each X_i is connected.*

Proof. (⇒) This follows from Prop 16.5, as $p_i : \prod_i X_i \longrightarrow X_i$ is surjective (this uses that all X_j are nonempty).

(⇐) Suppose each X_i is connected. By induction, it suffices to show that $X_1 \times X_2$ is connected. Pick any $z \in X_2$. We then have the embedding $X_1 \hookrightarrow X_1 \times X_2$ given by $x \mapsto (x, z)$. Since X_1 is connected, so is its image C in the product. Now for each $x_1 \in X_1$, we have an embedding $\iota_{x_1} : X_2 \hookrightarrow X_1 \times X_2$ given by $y \mapsto (x_1, y)$. Let $D_{x_1} = \iota_{x_1}(X_2) \cup C$. Note that each D is connected, being the overlapping union of two connected subsets. But we can write $X_1 \times X_2$ as the overlapping union of all of the D_{x_1}, so by the previous result the product is connected. ■

The following result is easy but useful.

Proposition 16.10. *Let $A \subseteq B \subseteq \overline{A}$ and suppose that A is connected. Then so is B.*

Proof. Exercise ■

Theorem 16.11. *Assume $X_i \neq \emptyset$ for all $i \in I$, where is I is arbitrary. Then $\prod_i X_i$ is connected if and only if each X_i is connected.*

Proof. As in the finite product case, it is immediate that if the product is connected, then so is each factor.

We sketch the other implication. We have already showed that each finite product is connected. Now let $(z_i) \in \prod_i X_i$. For each $j \in I$, write $D_j = p_j^{-1}(z_j) \subseteq \prod_i X_i$.

For each finite collection $j_1, \ldots, j_k \in I$, let

$$F_{j_1,\ldots,j_k} = \bigcap_{j \neq j_1,\ldots,j_k} D_j \subseteq \prod_i X_i.$$

Then $F_{j_1,\ldots,j_k} \cong X_{j_1} \times \cdots \times X_{j_k}$, so it follows that $F_{j_1,\ldots,j_k}$ is connected. Now $(z_i) \in F_{j_1,\ldots,j_k}$ for every such tuple, so it follows that

$$F = \bigcup F_{j_1,\ldots,j_k}$$

is connected.

It remains to show that F is dense in $\prod_i X_i$ (in other words, the closure of F is the whole product). Let

$$U = p_{j_1}^{-1}(U_{j_1}) \cap \cdots \cap p_{j_k}^{-1}(U_{j_k})$$

be a nonempty basis element. Then U meets $F_{j_1,\ldots,j_k}$, so U meets F. Since U was arbitrary, it follows that F must be dense. ■

Note that the above proof would not have worked with the box topology. We can show directly that $\mathbb{R}^{\mathbb{N}}$, equipped with the box topology, is not connected. Consider the subset $\mathcal{B} \subset \mathbb{R}^{\mathbb{N}}$ consisting of bounded sequences. If $(z_i) \in \mathcal{B}$, then $\prod_i (z_i - 1, z_i + 1)$ is a neighborhood of (z_i) in $\mathcal{B}$. On the other hand, if $(z_i) \notin \mathcal{B}$, the same formula gives a neighborhood consisting entirely of unbounded sequences. We conclude that $\mathcal{B}$ is a nontrivial clopen set in the box topology.