CLASS NOTES
MATH 551

BERTRAND GUILLOU

Contents

Wed, Aug. 23

Topology is the study of shapes. (The Greek meaning of the word is the study of places.) What kind of shapes? Many are familiar objects: a circle or triangle or square. From the point of view of topology, these are indistinguishable. Going up in dimension, we might want to study a sphere or box or a torus. Here, the sphere *is* topologically distinct from the torus. And neither of these is considered to be equivalent to the circle.

One standard way to distinguish the circle from the sphere is to see what happens when you remove two points. One case gives you two disjoint intervals, whereas the other gives you an (open) cylinder. The two intervals are disconnected, whereas the cylinder is not. This then implies that the circle and the sphere cannot be identified as topological spaces. This will be a standard line of approach for distinguishing two spaces: find a topological property that one space has and the other does not.

In fact, all of the above examples arise as **metric spaces**, but topology is quite a bit more general. For starters, a circle of radius 1 is the same as a circle of radius 123978632 from the eyes of topology. We will also see that there are many interesting spaces that can be obtained by modifying familiar metric spaces, but the resulting spaces cannot always be given a nice metric.

Part 1. **Metric Spaces**

1. Definitions

As we said, many examples that we care about are metric spaces, so we'll start by reviewing the theory of metric spaces.

Definition 1.1. A **metric space** is a pair (X, d), where X is a set and $d : X \times X \longrightarrow \mathbb{R}$ is a function (called a "metric") satisfying the following three properties:

(1) (Symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$
(2) (Positive-definite) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$
(3) (Triangle Inequality) $d(x, y) + d(y, z) \geq d(x, z)$ for all x, y, z in X.

Example 1.2. (1) $\mathbb{R}$ is a metric space, with $d(x, y) = |x - y|$.
(2) $\mathbb{R}^2$ is a metric space, with $d_{Euc}(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. This is called the standard, or Euclidean metric, on $\mathbb{R}^2$.
(3) (Euclidean metric) $\mathbb{R}^n$ similarly has a Euclidean metric, defined by

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

(4) (max metric) $\mathbb{R}^2$, with $d(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.
(5) (taxicab metric) $\mathbb{R}^2$, with $d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|$.
(6) (wheel metric) $\mathbb{R}^2$, where

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} d_{Euc}(\mathbf{x}, \mathbf{y}) & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ lie on a common line through } \mathbf{0} \\ d_{Euc}(\mathbf{x}, \mathbf{0}) + d_{Euc}(\mathbf{y}, \mathbf{0}) & \text{else} \end{cases}$$

Given a point x in a metric space X, we can consider those points "near to x".

Definition 1.3. Let (X, d) be a metric space and let $x \in X$. We define the (open) **ball of radius** r **around** x to be

$$B_r(x) = \{y \in X \mid d(x, y) < r\}.$$

Example 1.4. (1) In $\mathbb{R}$, with the usual metric, we have $B_r(x) = (x - r, x + r)$.
(2) In $\mathbb{R}^2$, with the standard metric, we have $B_r(\mathbf{x})$ is a disc of radius r, centered at $\mathbf{x}$.
(3) In $\mathbb{R}^n$, with the standard metric, we have $B_r(\mathbf{x})$ is an n-dimensional ball of radius r, centered at $\mathbf{x}$.
(4) In $\mathbb{R}^2$, with the max metric, $B_r(\mathbf{x})$ takes the form of a square, with sides of length $2r$, centered at $\mathbf{x}$.
(5) In $\mathbb{R}^2$, with the "taxicab" metric, $B_r(\mathbf{x})$ is a diamond, with sides of length $r\sqrt{2}$, centered at $\mathbf{x}$.
(6) In $\mathbb{R}^2$, with the "wheel" metric, there are two types of open balls.
   (a) If $r \leq d_{Euc}(\mathbf{x}, \mathbf{0})$, then $B_r(\mathbf{x})$ is an open interval, on the line through $\mathbf{x}$ and $\mathbf{0}$, centered at $\mathbf{x}$ of radius r.
   (b) But if $r > d_{Euc}(\mathbf{x}, \mathbf{0})$, write $\epsilon = r - d_{Euc}(\mathbf{x}, \mathbf{0})$. In this case, $B_r(\mathbf{x})$ is the union of (1) the open interval on the line through $\mathbf{x}$, centered at $\mathbf{x}$ and of radius r and (2) the Euclidean open ball centered at $\mathbf{0}$ of radius ϵ.

$\boxed{\textbf{Fri, Aug. 25}}$

1.1. **Cartesian Products.** In the definition of a metric space, we had a metric function $X \times X \longrightarrow \mathbb{R}$. Let's review: what is the set $X \times X$? More generally, what is $X \times Y$, when X and Y are sets? We know this as the set of ordered pairs

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

This is the usual definition of the **cartesian product** of two sets. One of the points of emphasis in this class will be not just objects or constructions but rather maps into/out of objects. With that in mind, given the cartesian product $X \times Y$, can we say anything about maps into or out of $X \times Y$?

The first thing to note is that there are two "natural" maps out of the product; namely, the projections. These are

$$p_X : X \times Y \longrightarrow X, \qquad p_X(x, y) = x$$

and

$$p_Y : X \times Y \longrightarrow Y \qquad p_Y(x, y) = y.$$

Now let's consider functions *into* $X \times Y$ from other, arbitrary, sets. Suppose that Z is a set. How would one specify a function $f : Z \longrightarrow X \times Y$? For each $z \in Z$, we would need to give a point $f(z) \in X \times Y$. This point can be described by listing its X and Y coordinates. Given that the projection p_X takes a point in the product and picks out its X-coordinate, it follows that the function f_X defined as the composition

$$Z \xrightarrow{f} X \times Y \xrightarrow{p_X} X$$

is the function of X-coordinates of the function f. We similarly get a function f_Y by using p_Y instead.

And the main point of this is that the function f contains the same information as the pair of functions f_X and f_Y. In other words, there is a bijective correspondence between functions f and pairs of functions (f_X, f_Y).

Proposition 1.5. *(Universal property of the cartesian product) Let X, Y, and Z be any sets. Suppose given functions $f_X : Z \longrightarrow X$ and $f_Y : Z \longrightarrow Y$. Then there exists a* **unique** *function $f : Z \longrightarrow X \times Y$ such that*

$$f_X = p_X \circ f, \qquad \text{and} \qquad f_Y = p_X \circ f.$$

$$\begin{tikzcd}
 & & X \\
Z \arrow[urr, bend left, "f_X"] \arrow[drr, bend right, "f_Y"'] \arrow[r, dashed, "\exists! f"] & X \times Y \arrow[ur, "p_X"'] \arrow[dr, "p_Y"] & \\
 & & Y
\end{tikzcd}$$

Furthermore, it turns out that the above property uniquely characterizes the cartesian product $X \times Y$, up to bijection. We called this a "Proposition", but there is nothing difficult about this, once you understand the statement. The major advance at this point is simply the reframing of a familiar concept. We will see later in the course why this is useful.

So far, it probably seems like we're taking something relatively simple and making it sound very complicated, but I promise this point of view will pay off down the line!

2. Continuous Functions

As we already said, we will promote the viewpoint that it is not just objects that are important, but also maps. We have introduced the concept of a metric space, so we should then ask "What are maps between metric spaces"?

The strictest answer is what is known as an **isometry**: a function $f : X \longrightarrow Y$ such that $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$ for all pairs of points x_1 and x_2 in X. This is a perfectly fine answer in many regards, but for our purposes, it will be too restrictive. For instance, what are all isometries $\mathbb{R} \longrightarrow \mathbb{R}$?

We will prefer to study the more general class of **continuous** functions.

Definition 2.1. A function $f : X \longrightarrow Y$ between metric spaces is **continuous** if for every $x \in X$ and for every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $x' \in B_\delta(x)$, then $f(x') \in B_\varepsilon(f(x))$.

This is the standard definition, taken straight from Calc I and written in the language of metric spaces. However, it is not always the most convenient formulation.

Proposition 2.2. *Let $f : X \longrightarrow Y$ be a function between metric spaces. The following are equivalent:*

(1) *f is continuous*
(2) *for every $x \in X$ and for every $\varepsilon > 0$, there is a $\delta > 0$ such that*
$$B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x)))$$
(3) *For every $y \in Y$ and $\epsilon > 0$ and $x \in X$, if $f(x) \in B_\epsilon(y)$, then there exists a $\delta > 0$ such that*
$$B_\delta(x) \subseteq f^{-1}(B_\epsilon(y))$$
(4) *For every $y \in Y$ and $\epsilon > 0$ and $x \in X$, if $x \in f^{-1}(B_\epsilon(y))$, then there exists a $\delta > 0$ such that*
$$B_\delta(x) \subseteq f^{-1}(B_\epsilon(y))$$

2.1. **Open Sets.** The property that $f^{-1}(B_\epsilon(y))$ satisfies in condition (4) is important, and we give it a name:

Definition 2.3. Let $U \subseteq X$ be a subset. We say that U is **open** in X if whenever $x \in U$, then there exists a $\delta > 0$ such that $B_\delta(x) \subseteq U$.

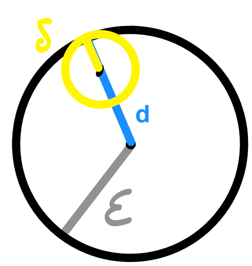
With this language at hand, we can restate condition (4) above as

$(4')$ For every $y \in Y$ and $\epsilon > 0$, $f^{-1}(B_\epsilon(y))$ is open in X.

The language suggests that an open ball should count as an open set, and this is indeed true.

Proposition 2.4. *Let $c \in X$ and $\epsilon > 0$. Then $B_\epsilon(c)$ is open in X.*

Proof. Suppose $x \in B_\epsilon(c)$. This means that $d(x, c) < \epsilon$. Write d for this distance. Let
$$\delta = \epsilon - d.$$
We claim that this is the desired δ. For suppose that $u \in B_\delta(x)$. Then
$$d(u, c) \leq d(u, x) + d(x, c) < \delta + d = \epsilon.$$
∎



Ok, so the notion of open set is closely related to that of open ball: every open ball is an open set, and every open set is required to contain a number of these open balls. Even better, we have the following result:

Proposition 2.5. *A subset $U \subseteq X$ is open if and only if it can be expressed as a union of open balls.*

Proof. Suppose U is open, and let $x \in U$. By definition, there exists $\delta_x > 0$ with $B_{\delta_x}(x) \subseteq U$. Since this is true for every $x \in U$, we have
$$\bigcup_{x \in U} B_{\delta_x}(x) \subseteq U.$$
But every $x \in U$ is contained in the union, so clearly U must also be contained in the union. It follows that
$$\bigcup_{x \in U} B_{\delta_x}(x) = U.$$

Now suppose, on the other hand, that $U = \bigcup_\alpha B_{\delta_\alpha}(x_\alpha)$. We wish to show that U is open. Well, suppose $u \in U$. Since U is expressed as a union, this implies that $u \in B_{\delta_\alpha}(x_\alpha)$ for some α. This ball is contained in U by the definition of U, so we are done. ■

Corollary 2.6. *Any union of open subsets of X is open.*

With this description of open sets in hand, we give what is often the most useful characterization of continuous maps.

Proposition 2.7. *Let $f : X \longrightarrow Y$ be a function between metric spaces. The following are equivalent:*

(1) *f is continuous*
(5) *For every open subset $V \subseteq Y$, the preimage $f^{-1}(V)$ is open in X.*

Proof. It is clear that (5) implies ($4'$), which is equivalent to (1) by Prop 2.2. Now assume (1), or, equivalently, ($4'$). Let $V \subseteq Y$ be open. By the previous result, V is a union of balls, and by ($4'$) we know that the preimage of each ball is open. Using Corollary 2.6, it follows that $f^{-1}(V)$ is open. ■

Mon, Aug. 28

2.2. **Convergence.** In calculus, we are also used to thinking of continuity in terms of convergence of sequences. Recall that a sequence (x_n) in X **converges** to x if for every $\epsilon > 0$ there exists N such that for all $n > N$, we have $x_n \in B_\epsilon(x)$. We say that a "tail" of the sequence is contained in the ball around x.

Proposition 2.8. *The sequence (x_n) converges to x if and only if for every open set U containing x, some tail of (x_n) lies in U.*

Proof. Exercise. ■

Proposition 2.9. *Let $f : X \longrightarrow Y$ be a function between metric spaces. The following are equivalent:*

(1) *f is continuous*
(6) *For every convergent sequence $(x_n) \to x$ in X, the sequence $(f(x_n))$ converges to $f(x)$ in Y.*

Proof. This is on HW1. ■

This finishes our discussion of continuity.

What constructions can we make with metric spaces?

3. Products

Let's start with a product. That is, if (X, d_X) and (Y, d_Y) are metric spaces, is there a good notion of the product metric space? We would want to have continuous "projection" maps to each of X and Y, and we would want it to be true that to define a continuous map from some metric space Z to the product, it is enough to specify continuous maps to each of X and Y. By thinking about the case in which Z has a discrete metric, one can see that the underlying set of the product metric space would need to be the cartesian product $X \times Y$. The only question is whether or not there is a sensible metric to define.

Recall that we discussed several metrics on $\mathbb{R}^2$, including the standard one, the max metric, and the taxicab metric. There, we used that $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ as an underlying set, and we combined the metrics on each copy of $\mathbb{R}$ to get a metric on $\mathbb{R}^2$. We can use the same idea here to get three different metrics on $X \times Y$, and these will all produce a metric space satisfying the right property to be a product.

For convenience, let's pick the max metric on $X \times Y$. To show that the projection $p_X : X \times Y \longrightarrow X$ is continuous, it is enough to show that each $p_X^{-1}(B_\epsilon(x))$ is open. But it is simple to show that

$$p_X^{-1}(B_\epsilon(x)) = B_\epsilon(x) \times Y$$

is open using the max metric. The same argument shows that p_Y is continuous.

Now suppose that Z is another metric space with continuous maps $f_X, f_Y : Z \rightrightarrows X, Y$. We define $f = (f_X, f_Y)$ coordinate-wise as before, and it only remains to show that it is continuous. Consider a ball $B_\epsilon(x, y) \subset X \times Y$. Under the max metric, this ball can be rewritten as

$$B_\epsilon(x, y) = B_\epsilon(x) \times B_\epsilon(y),$$

so that

$$f^{-1}(B_\epsilon(x, y)) = f^{-1}\big(B_\epsilon(x) \times B_\epsilon(y)\big) = f_X^{-1}(B_\epsilon(x)) \cap f_Y^{-1}(B_\epsilon(y)).$$

By a problem from HW1, this is open, showing that f is continuous.

Question 3.1. *What if we have infinitely many metric spaces X_i, as i ranges through some indexing set $\mathcal{I}$? Can we make sense of a product $\prod_{i \in \mathcal{I}} X_i$ as a metric space?*

Wed, Aug. 30

Last time, we discussed the product of metric spaces. We were guided to our definition by considering the "universal property" that the product should satisfy. But why is this enough? Couldn't there be another metric space that shares the same universal property? It turns out that the answer is no: any two metric spaces that share the same universal property are essentially "the same". We will come back to this point when we discuss products in the context of topological spaces.

4. Function spaces

Another important construction is that of a space of functions. That is, if X and Y are metric spaces, one can consider the set of all continuous functions $f : X \longrightarrow Y$. Is there a good way to think of this as a metric space? For example, as a set $\mathbb{R}^2$ is the same as the collection of functions $\{1, 2\} \longrightarrow \mathbb{R}$. More generally, we could consider functions $\{1, \dots, n\} \longrightarrow Y$ or even $\mathbb{N} \longrightarrow Y$ (i.e. sequences).

Again, we first step back and consider what happens in the simpler world of sets. We will write $F(Y, Z)$ for the set of functions $Y \longrightarrow Z$, also denoted Z^Y. Then given a function $\varphi : X \times Y \longrightarrow Z$, we can define a function $\hat{\varphi} : X \longrightarrow F(Y, Z)$ by the formula

$$\hat{\varphi}(x)(y) = \varphi(x, y).$$

Conversely, given a function $\hat{\varphi}$, the equation above defines the function φ. In other words, we have a bijection

$$F(X \times Y, Z) \cong F(X, F(Y, Z)).$$

We might ask if a similar story exists in the world of metric spaces.

Of the metrics we discussed on $\mathbb{R}^2$, the max metric generalizes most easily to give a metric on $Y^\infty = Y^{\mathbb{N}}$. We provisionally define the **sup metric** on the set of sequences in Y by

$$d_{\sup}((y_n), (z_n)) = \sup_n \{d_Y(y_n, z_n)\}.$$

Without any further restrictions, there is no reason that this supremum should always exist. If Y is a bounded metric space, or if we only consider bounded sequences, then we are OK. Another option is to arbitrarily truncate the metric.

Lemma 4.1. *Let (Y, d) be a metric space. Define the resulting bounded metric $\overline{d}$ on Y by*

$$\overline{d}(y, z) = \min\{d(y, z), 1\}.$$

This is a metric, and the open sets determined by $\overline{d}$ are precisely the open sets determined by d.

We now redefine the sup metric on Y^∞ to be

$$d_{\sup}((y_n), (z_n)) = \sup_n \{\overline{d_Y}(y_n, z_n)\}.$$

Now the supremum always exists, so that we get a well-defined metric. The same definition works to give a metric on the set of continuous functions $X \longrightarrow Y$. We define the sup metric on the set $\mathcal{C}(X, Y)$ of continuous functions to be

$$d_{\sup}(f, g) = \sup_{x \in X} \{\overline{d_Y}(f(x), g(x))\}.$$

This is also called the **uniform metric**, for the following reason.

Proposition 4.2. *Let (f_n) be a sequence in $\mathcal{C}(X,Y)$. Then $(f_n) \to f$ in the uniform metric on $\mathcal{C}(X,Y)$ if and only if $(f_n) \to f$ uniformly.*

Given a function $f \in \mathcal{C}(X,Y)$ and a point $x \in X$, one can evaluate the function to get $f(x) \in Y$. In other words, we have an evaluation function

$$\mathrm{eval} : \mathcal{C}(X,Y) \times X \longrightarrow Y.$$

Proposition 4.3. *Consider $\mathcal{C}(X,Y) \times X$ as a metric space using the max metric. Then* eval *is continuous.*

Proof. By Proposition 2.9, to determine if a function between metric spaces is continuous, it suffices to check that it takes convergent sequences to convergent sequences. Suppose that $(f_n, x_n) \to (f, x)$. We wish to show that

$$\mathrm{eval}(f_n, x_n) = f_n(x_n) \to \mathrm{eval}(f, x) = f(x).$$

Since $(f_n, x_n) \to (f, x)$, it follows that $f_n \to f$ and $x_n \to x$ (since the projections are continuous).

Let $\varepsilon > 0$. Then there exists N_1 such that if $n > N_1$ then $d_{\sup}(f_n, f) < \varepsilon/2$. By the definition of the sup metric, this implies that $d_Y(f_n(x_n), f(x_n)) < \varepsilon/2$. But now f is continuous, so there exists N_2 such that if $n > N_2$ then $d_Y(f(x_n), f(x)) < \varepsilon/2$. Putting these together and using the triangle inequality, if $n > N_3 = \max\{N_1, N_2\}$ then $d_Y(f_n(x_n), f(x)) < \varepsilon$. ∎

$\boxed{\textbf{Fri, Sept. 1}}$

Proposition 4.4. *Suppose $\varphi : X \times Y \longrightarrow Z$ is continuous. For each $x \in X$, define $\hat{\varphi}(x) : Y \longrightarrow Z$ by $\hat{\varphi}(x)(y) = \varphi(x, y)$. The function $\hat{\varphi}(x)$ is continuous.*

Proof. This could certainly be done directly, using convergence of sequences to test for continuity. Here is another way to do it, using the universal property of products.

Note that $\hat{\varphi}(x)$ can be written as the composition $Y \xrightarrow{i_x} X \times Y \xrightarrow{\varphi} Z$. By assumption, φ is continuous, so it suffices to know that $i_x : Y \to X \times Y$ is continuous. But recall that continuous maps into a product correspond precisely to a pair of continuous maps into each factor. The pair of maps here is the constant map $Y \longrightarrow X$ at x and the identity map $Y \longrightarrow Y$. The identity map is clearly continuous, and the constant map is continuous since if $U \subseteq X$ is open, then the preimage under the constant map is either (1) all of Y if $x \in U$ or (2) empty if $x \notin U$. So it follows that i_x is continuous. ∎

We are headed to the universal property of the mapping space. Keeping the notation from above, given a continuous function

$$\varphi : X \times Y \longrightarrow Z,$$

we get a function

$$\hat{\varphi} : X \longrightarrow \mathcal{C}(Y, Z).$$

Conversely, given the function $\hat{\varphi}$, we define φ by

$$\varphi(x, y) = \hat{\varphi}(x)(y).$$

Proposition 4.5. *The function φ above is continuous if $\hat{\varphi}$ is continuous.*

Proof. On homework 2. ∎

Of course, we would like this to be an if and only if, but that is only true under additional hypotheses (like Y compact, for instance.) Another way to state the if-and-only-if version of this proposition is that we get a bijection

$$\mathcal{C}(X \times Y, Z) \cong \mathcal{C}(X, \mathcal{C}(Y, Z)).$$

For those who have seen the $(\otimes, \mathrm{Hom})$ adjunction in algebra, this is completely analogous.

5. Quotients

Another (very) important construction that we will discuss when we move on to topological spaces is that of a quotient, or identification space. A standard example is the identification, on the unit interval $[0,1]$, of the two endpoints. Glueing these together gives a circle S^1, and the surjective continuous map

$$e^{2\pi ix} : [0,1] \longrightarrow S^1$$

is called the quotient map. Here the desired universal property is that if $f : [0,1] \longrightarrow Y$ is a continuous map to another metric space such that $f(0) = f(1)$, then the map f should "factor" through the quotient. Quotients become quite complicated to express in the world of metric spaces.

Part 2. Topological Spaces

Now that we have spent some time with metric spaces, let's turn to the more general world of topological spaces.

6. Definitions

Definition 6.1. A **topological space** is a set X with a collection of subsets $\mathcal{T}$ of X such that

(1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$
(2) If $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$
(3) If $U_i \in \mathcal{T}$ for all i in some index set I, then $\bigcup_{i\in I} U_i \in \mathcal{T}$.

The collection $\mathcal{T}$ is called the **topology** on X, and the elements of $\mathcal{T}$ are referred to as the "open sets" in the topology.

Example 6.2. (1) (Metric topology) Any metric space is a topological space, where $\mathcal{T}$ is the collection of metric open sets

(2) (Discrete topology) In the discrete topology, *every* subset is open. We already saw the discrete metric on any set, so in fact this is an example of a metric topology as well.
(3) (Trivial topology) In the trivial topology, $\mathcal{T} = \{\emptyset, X\}$. That is, $\emptyset$ and X are the only empty sets. This topology does not come from a metric (unless X has fewer than two points).
(4) It is simple to write down various topologies on a finite set. For example, on the set

$$X = \{1, 2\},$$

there are 4 possible topologies. In addition to the trivial and discrete topologies, there is also

$$\mathcal{T}_1 = \{\emptyset, \{1\}, X\}$$

and

$$\mathcal{T}_2 = \{\emptyset, \{2\}, X\}.$$

(5) There are many possible topologies on $X = \{1, 2, 3\}$. But not every collection of subsets will give a topology. For instance,

$$\{\emptyset, \{1, 2\}, \{1, 3\}, X\}$$

would not be a topology, since it is not closed under intersection.

Wed, Sept. 6

At the end of class on Friday, we introduced the notion of a topology, and I asked you to think about how many possible topologies there are on a 3-element set. The answer is . . . 29. The next few answers for the number of topologies on a set of size n are[1]: 355 ($n = 4$), 6942 ($n = 5$), 209527 ($n = 6$). But there is no known formula for answer in general.

6.1. **Bases.** When working with metric spaces, we saw that the topology was determined by the open balls. Namely, an open set was precisely a subset that could be written as a union of balls. In many topologies, there is an analogue of these basic open sets.

Definition 6.3. A **basis** for a topology on X is a collection $\mathcal{B}$ of subsets such that

(1) (Covering property) Every point of x lies in at least one basis element
(2) (Intersection property) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a third basis element B_3 such that
$$x \in B_3 \subseteq B_1 \cap B_2.$$

A basis $\mathcal{B}$ defines a topology $\mathcal{T}_{\mathcal{B}}$ by declaring the open sets to be the unions of (arbitrarily many) basis elements.

Proposition 6.4. *Given a basis $\mathcal{B}$, the collection $\mathcal{T}_{\mathcal{B}}$ is a topology.*

Proof. It is clear that open sets are closed under unions. The emptyset is a union of no basis elements, so it is open. The set X is open by the covering property: the union of all basis elements is X. Finally, we check that the intersection of two open sets is open. Let U_1 and U_2 be open. Then
$$U_1 = \bigcup_{\alpha \in A} B_\alpha, \qquad U_2 = \bigcup_{\delta \in \Delta} B_\delta.$$
We want to show that $U_1 \cap U_2$ is open. Now
$$U_1 \cap U_2 = \left(\bigcup_{\alpha \in A} B_\alpha\right) \cap \left(\bigcup_{\delta \in \Delta} B_\delta\right) = \bigcup_{\alpha \in A, \delta \in \Delta} B_\alpha \cap B_\delta.$$
It remains to show that $B_\alpha \cap B_\delta$ is open. By the intersection property of a basis, for each $x \in B_\alpha \cap B_\delta$, there is some B_x with
$$x \in B_x \subseteq B_\alpha \cap B_\delta.$$
It follows that
$$B_\alpha \cap B_\delta = \bigcup_{x \in B\alpha \cap B_\delta} B_x,$$
so we are done. ∎

Example 6.5. We have already seen that metric balls form a basis for the metric topology. In the case of the discrete metric, one can take the balls with radius 1/2, which are exactly the singleton sets.

Example 6.6. For a truly new example, we take as basis on $\mathbb{R}$, the half-open intervals $[a, b)$. The resulting topology is known as the **lower limit topology** on $\mathbb{R}$.

How is this related to the usual topology on $\mathbb{R}$? Well, any open interval (a, b) can be written as a union of half-open intervals. However, the $[a, b)$ are certainly not open in the usual topology. This says that $\mathcal{T}_{\text{standard}} \subseteq \mathcal{T}_{\ell\ell}$. The lower limit topology has more open sets than the usual topology. When one topology on a set has more open sets than another, we say it is **finer**. So the lower limit

[1]These are taken from the On-Line Encyclopedia of Integer Sequences.

topology is *finer* than the usual topology on $\mathbb{R}$, and the usual topology is *coarser* than the lower limit topology.

On any set X, the discrete topology is the finest, whereas the trivial topology is the coarsest.

When a topology is generated by a basis, there is a convenient criterion for open sets.

Proposition 6.7. *(Local criterion for open sets) Let $\mathcal{T}_{\mathcal{B}}$ be a topology on X generated by a basis $\mathcal{B}$. Then a set $U \subseteq X$ is open if and only if, for each $x \in U$, there is a basis element $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U$.*

Proof. ($\Rightarrow$) By definition of $\mathcal{T}_{\mathcal{B}}$, the set U is a union of basis elements, so any $x \in U$ must be contained in one of these.

($\Leftarrow$) We can write $U = \bigcup_{x \in U} B_x$. ∎

This is a good time to introduce a convenient piece of terminology: given a point x of a space X, a **neighborhood** N of x in X is a subset of X containing some open set U with $x \in U \subseteq N$. Often, we will take our neighborhoods to themselves be open.

7. Continuous Functions

Given our discussion of continuous maps between metric spaces, it should be clear what the right notion is for maps between topological spaces.

Definition 7.1. A function $f : X \longrightarrow Y$ between topological spaces is said to be **continuous** if for every open subset $V \subseteq Y$, the preimage $f^{-1}(V)$ is open in X.

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Example 7.2. Let $X = \{1, 2\}$ with topology $\mathcal{T}_X = \{\emptyset, \{1\}, X\}$ and let $Y = \{1, 2, 3\}$ with topology $\mathcal{T}_Y = \{\emptyset, \{2\}, \{3\}, \{2, 3\}, Y\}$. Which functions $X \longrightarrow Y$ are continuous?

Let's start with the open set $\{2\} \subseteq Y$. The preimage must be open, so it can either be $\emptyset$ or $\{1\}$ or X. If the preimage is X, the function is constant at 2, which is continuous.

Suppose the preimage is $\emptyset$. Then the preimage of $\{3\}$ can be either $\emptyset$ or $\{1\}$ or X. If it is $\emptyset$, we are looking at the constant function at 1, which is continuous. If $f^{-1}(3) = X$, then f is constant at 3, which is continuous. Finally, if $f^{-1}(3) = \{1\}$, then f must be the continuous function $f(1) = 3$, $f(2) = 1$.

Finally, suppose $f^{-1}(2) = \{1\}$. Then $f^{-1}(3)$ can't be $\{1\}$ or X, so the only possible continuous f has $f^{-1}(3) = \emptyset$, so that we must have $f(1) = 2$ and $f(2) = 1$.

By the way, we asserted above that constant functions are continuous. We proved this before (top of page 7) for metric spaces, but the proof given there applies verbatim to general topological spaces.

Proposition 7.3. *Suppose $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are continuous. Then so is their composition $g \circ f : X \longrightarrow Z$.*

Proof. Let $V \subseteq Z$ be open. Then

$$\begin{aligned}(g \circ f)^{-1}(V) &= \{x \in X \mid (g \circ f)(x) \in V\} = \{x \in X \mid g(f(x)) \in V\} \\ &= \{x \in X \mid f(x) \in g^{-1}(V)\} = \{x \in X \mid x \in f^{-1}(g^{-1}(V))\} = f^{-1}(g^{-1}(V)).\end{aligned}$$

Now g is continuous, so $g^{-1}(V)$ is open in Y, and f is continuous, so $f^{-1}(g^{-1}(V))$ is open in X. ∎

8. Subspace Topology

Another construction we can consider with continuous functions is the idea of restricting a continuous function to a subset. For instance, the natural logarithm is a nice continuous function $\ln : (0, \infty) \longrightarrow \mathbb{R}$, but we also get a nice continuous function by considering the logarithm only on $[1, \infty)$. To have this discussion here, we should think about how a subset of a space becomes a space in its own right.

Definition 8.1. Let X be a space and let $A \subseteq X$ be a subset. We define the **subspace topology** on A by saying that $V \subseteq A$ is open if and only if there exists some open $U \subseteq X$ with $U \cap A = V$.

Note that the open set $U \subseteq X$ is certainly not unique.

Example 8.2.
(1) Let $A = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$. Then the subspace topology on $A \cong \mathbb{R}$ is the usual topology on $\mathbb{R}$. Indeed, consider the usual basis for $\mathbb{R}^2$ consisting of open disks. Intersecting these with A gives open intervals. In general, intersecting a basis for X with a subset A gives a basis for A, and here we clearly get the usual basis for the standard topology. The same would be true if we started with max-metric basis (consisting of open rectangles).
(2) Let $A = (0, 1) \subseteq X = \mathbb{R}$. We claim that $V \subseteq A$ is open in the subset topology if and only if V is open as a subset of $\mathbb{R}$. Indeed, suppose that V is open in A. Then $V = U \cap (0, 1)$ for some open U in $\mathbb{R}$. But now both U and $(0, 1)$ are open in $\mathbb{R}$, so it follows that their intersection is as well. The converse is clear.
   Note that this statement fails for the previous example. $(0, 1) \times \{0\}$ is open in A there but not open in $\mathbb{R}^2$.
(3) Let $A = (0, 1]$. Then, in the subspace topology on A, every interval $(a, 1]$, with $a < 1$ is an open set. A basis for this topology on A consists in the (a, b) with $0 \leq a < b < 1$ and the $(a, 1]$ with $0 \leq a < 1$.
(4) Let $A = (0, 1) \cup \{2\}$. Then the singleton $\{2\}$ is an open subset of A! A basis consists of the (a, b) with $0 \leq a < b \leq 1$ and the singleton $\{2\}$.

Given a subset $A \subseteq X$, there is always the inclusion function $\iota_A : A \longrightarrow X$ defined by $\iota_A(a) = a$.

Proposition 8.3. *Given a subset $A \subseteq X$ of a topological space, the inclusion ι_A is continuous. Moreover, the subspace topology on A is the coarsest topology which makes this true.*

Proof. Suppose that $U \subseteq X$ is open. Then $\iota_A^{-1}(U) = U \cap A$ is open in A by the definition of the subspace topology.

To see that this is the coarsest such topology, suppose that $\mathcal{T}'$ is a topology which makes the inclusion $\iota_A : A \longrightarrow X$. We wish to show that $\mathcal{T}'$ is finer than the subspace topology, meaning that $\mathcal{T}_A \subseteq \mathcal{T}'$, where $\mathcal{T}_A$ is the subspace topology. So let V be open in $\mathcal{T}_A$. This means there exists $U \subseteq X$ open such that $V = U \cap A = \iota_A^{-1}(U)$. Since ι_A is continuous according to $\mathcal{T}'$, it follows that V is open in $\mathcal{T}'$. ■

Getting back to our motivational question, suppose that $f : X \longrightarrow Y$ is continuous and let $A \subseteq X$ be a subset. We define the restriction of f to A, denoted $f_{|A}$, by

$$f_{|A} : A \longrightarrow Y, \qquad f_{|A}(a) = f(a).$$

Proposition 8.4. *Let $f : X \longrightarrow Y$ be continuous and suppose that $A \subseteq X$ is a subset. Then the restriction $f_{|A} : A \longrightarrow Y$ is continuous.*

Proof. This is just the composition $f_{|A} = f \circ \iota_A$. ■

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9. Closed Sets

So far, we only discussed the notion of open set, but there is also the complementary notion of closed set.

Definition 9.1. Let X be a space. We say a subset $W \subseteq X$ is **closed** if the complement $X \setminus W$ is open.

Note that, despite what the name may suggest, closed does *not* mean "not open". For instance, the empty set is always both open (required for any topology) and closed (because the complement, X must be open). Similarly, there are many examples of sets that are neither open nor closed (for example, the interval $[0, 1)$ in the usual topology on $\mathbb{R}$).

Proposition 9.2. *Let X be a space.*

1. *$\emptyset$ and X are both closed in X*
2. *If W_1, W_2 are closed, then $W_1 \cup W_2$ is also closed*
3. *If W_i are closed for all i in some index set I, then $\bigcap_{i \in I} W_i$ is also closed.*

Proof. We prove (2). The point is that

$$X \setminus (W_1 \cup W_2) = (X \setminus W_1) \cap (X \setminus W_2).$$

This equality is known as one of the DeMorgan Laws ∎

Example 9.3. Consider $\mathbb{R}_{\ell\ell}$, the real line equipped with the lower-limit topology. (Example 6.6). There, a half-open interval $[a, b)$ was declared to be open. It then follows that intervals of the form $(-\infty, b)$ and $[a, \infty)$ are open. But this then implies that $[a, b)$ is *closed* since its complement is the open set $(-\infty, a) \cup [b, \infty)$.

Not only does a topology give rise to a collection of closed sets satisfying the above properties, but one can also define a topology by specifying a list of closed sets satisfying the above properties.

Similarly, we can use closed sets to determine continuity.

Proposition 9.4. *Let $f : X \longrightarrow Y$. Then f is continuous if and only if the preimage of every closed set in Y is closed in X.*

Example 9.5. The "distance from the origin function" $d : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is continuous (follows from HW 2). Since $\{1\} \subseteq \mathbb{R}$ is closed, it follows that the sphere $S^2 = d^{-1}(1)$ is closed in $\mathbb{R}^3$. More generally, S^{n-1} is closed in $\mathbb{R}^n$.

Example 9.6. Let X be any metric space, let $x \in X$, and let $r > 0$. Then the ball

$$B_{\leq r}(x) = \{y \in X \mid d(x, y) \leq r\}$$

is closed in X.

Remark 9.7. Note that some authors use the notation $\overline{B_r(x)}$ for the closed ball. This is a bad choice of notation, since it suggests that the closure of the open ball is the closed ball. But this is not always true! For instance, consider a set (with more than one point) equipped with the discrete metric. Then $B_1(x) = \{x\}$ is already closed, so it is its own closure. On the other hand, $B_{\leq 1}(x) = X$.

Consider the half-open interval $[a, b)$. It is neither open nor closed, in the usual topology. Nevertheless, there is a closely associated closed set, $[a, b]$. Similarly, there is a closely associated open set, (a, b). Notice the containments

$$(a, b) \subseteq [a, b) \subseteq [a, b].$$

It turns out that this picture generalizes.

9.1. **Closure and Interior.** Let's start with the closed set. In the example above, $[a, b]$ is the *smallest closed set containing* $[a, b)$. Why should we expect such a smallest closed set to exist in general? Recall that if we intersect arbitrarily many closed sets, we are left with a closed set.

Definition 9.8. Let $A \subseteq X$ be a subset of a topological space. We define the **closure of A in X** to be

$$\overline{A} = \bigcap_{A \subset B \text{ closed}} B.$$

Dually, we have $(a, b) \subset [a, b)$, and (a, b) is the largest open set contained in $[a, b)$.

Definition 9.9. Let $A \subseteq X$ be a subset of a topological space. We define the **interior of A in X** to be

$$\mathrm{Int}(A) = \bigcup_{A \supset U \text{ open}} U.$$

The difference of these two constructions is called the **boundary of A in X**, defined as

$$\partial A = \overline{A} \setminus \mathrm{Int}(A).$$

Example 9.10.
(1) From what we have already said, it follows that $\partial[a, b) = \{a, b\}$.
(2) Let $A = \{1/n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$. Then A is not open, since no neighborhood of any $1/n$ is contained in A. This also shows that $\mathrm{Int}(A) = \emptyset$. But neither is A closed, because no neighborhood of 0 is contained in the complement of A. This implies that $0 \in \overline{A}$, and it turns out that $\overline{A} = A \cup \{0\}$. Thus $\partial A = \overline{A} = A \cup \{0\}$.
(3) Let $\mathbb{Q} \subseteq \mathbb{R}$. Similarly to the example above, $\mathrm{Int}(\mathbb{Q}) = \emptyset$. But since $\mathbb{R} \setminus \mathbb{Q}$ does not entirely contain any open intervals, it follows that $\overline{\mathbb{Q}} = \mathbb{R}$. (A subset $A \subseteq X$ is said to be **dense** in X if $\overline{A} = X$.) Thus $\partial \mathbb{Q} = \mathbb{R} \setminus \emptyset = \mathbb{R}$.
(4) Let's turn again to $\mathbb{R}_{\ell\ell}$. We saw that $[0, 1)$ was already closed. What about $(0, 1]$? Since $[0, 1]$ is closed in the usual topology, this must be closed in $\mathbb{R}_{\ell\ell}$ as well. (Recall that the topology on $\mathbb{R}_{\ell\ell}$ is finer than the standard one). It follows that $(0, 1]$ is either already closed, or its closure is $[0, 1]$. We can ask, dually, whether the complement is open. But $(-\infty, 0] \cup (1, \infty)$ is not open since it does not contain any neighborhoods of 0. It follows that $\overline{(0, 1]} = [0, 1]$ in $\mathbb{R}_{\ell\ell}$.

There is a convenient characterization of the closure, which we were implicitly using above.

Proposition 9.11 (Neighborhood criterion)**.** *Let $A \subseteq X$. Then $x \in \overline{A}$ if and only if every neighborhood of x meets A.*

Proof. ($\Rightarrow$) Suppose $x \in \overline{A}$. Then $x \in B$ for all closed sets B containing A. Let N be a neighborhood of x. Without loss of generality, we may suppose N is open. Now $X \setminus N$ is closed but $x \notin X \setminus N$, so this set cannot contain A. This means precisely that $N \cap A \neq \emptyset$.

($\Leftarrow$) Suppose every neighborhood of x meets A. Let $A \subset B$, where B is closed in X. Now $U = X \setminus B$ is an open set not meeting A, so it cannot be a neighborhood of x. This must mean that $x \notin X \setminus B$, or in other words $x \in B$. Since B was arbitrary, it follows that x lies in every such B. ■

Wed, Sept. 13

10. Convergence

In our earlier discussion of metric spaces, we considered convergence of sequences and how this characterized continuity. This is one statement from the theory of metric spaces that will not carry over to the generality of topological spaces.

Definition 10.1. We say that a sequence x_n in X converges to x in X if every neighborhood of x contains a tail of (x_n).

The following result follows immediately from the previous characterization of the closure.

Proposition 10.2. *Let (a_n) be a sequence in $A \subseteq X$ and suppose that $a_n \to x \in X$. Then $x \in \overline{A}$.*

Proof. We use the neighborhood criterion. Thus let U be a neighborhood of x. Since $a_n \to x$, a tail of (a_n) lies in U. It follows that $U \cap A \neq \emptyset$, so that $x \in \overline{A}$. ■

However, the converse is not true in a general topological space. (The fact that these are equivalent in a metric space is the **sequence lemma**, Proposition 10.12.)

Example 10.3. Consider $\mathbb{R}$ equipped with the *cocountable* topology. Recall that this means that the nonempty open subsets are the cocountable ones.

Lemma 10.4. *Suppose that $x_n \to x$ in the cocountable topology on $\mathbb{R}$. Then (x_n) is eventually constant.*

Proof. Write B for the set
$$B = \{x_n \mid x_n \neq x\}.$$
Certainly B is countable, so it is closed. By construction, $x \notin B$, so $N = X \setminus B$ is an open neighborhood of x. But $x_n \to x$, so a tail of this sequence must lie in N. Since $\{x_n\} \cap N = \{x\}$, this means that a tail of this sequence is constant, in other words, the sequence is eventually constant. ■

Now consider $A = \mathbb{R} \setminus \{0\} \subseteq \mathbb{R}$ in the cocountable topology. A is not closed since the only closed proper subsets are the countable ones. It follows that A must be dense in $\mathbb{R}$. However, no sequence in A can converge to 0 since a convergent sequence must be eventually constant.

Similarly, we cannot use convergence of sequences to test for continuity in general topological spaces. For instance, consider the identity map
$$\mathrm{id} : \mathbb{R}_{\text{cocountable}} \longrightarrow \mathbb{R}_{\text{standard}},$$
where the domain is given the cocountable topology and the codomain is given the usual topology. This is not continuous, since the interval $(0, 1)$ is open in $\mathbb{R}_{\text{standard}}$ but not in $\mathbb{R}_{\text{cocountable}}$. On the other hand, the identity function takes convergent sequences in $\mathbb{R}_{\text{cocountable}}$, which are necessarily eventually constant, to convergent sequences in $\mathbb{R}_{\text{standard}}$. This follows from the following result, which you proved on HW1.

Proposition 10.5. *Let $f : X \longrightarrow Y$ be continuous. If $x_n \to x$ in X then $f(x_n) \to f(x)$ in Y.*

Proof. Suppose $x_n \to x$. Let V be an open neighborhood of $f(x)$. Then, since f is continuous, $f^{-1}(V)$ is an open neighborhood of x. Thus some tail of (x_n) lies in $f^{-1}(V)$, which means that the corresponding tail of $(f(x_n))$ lies in U. ■

However, all hope is not lost, since the following is true.

Proposition 10.6. *Let $f : X \longrightarrow Y$. Then f is continuous if and only if*
$$f(\overline{A}) \subseteq \overline{f(A)}$$
for every subset $A \subseteq X$.

Proof. ($\Rightarrow$) Assume f is continuous. Since $\overline{f(A)}$ is the intersection of *all* closed sets containing $f(A)$, it suffices to show that if B is such a closed set, then $f(\overline{A}) \subseteq B$. Well, $f(A) \subseteq B$, so

$$A = f^{-1}(f(A)) \subseteq f^{-1}(B).$$

Now f is continuous and B is closed, so by definition of the closure, we must have

$$\overline{A} \subseteq f^{-1}(B).$$

Applying f then gives $f(\overline{A}) \subseteq f(f^{-1}(B)) \subseteq B$.

($\Leftarrow$) Suppose that the above subset inclusion holds, and let $B \subseteq Y$ be closed. Let $A = f^{-1}(B)$. We wish to show that A is closed, i.e. that $\overline{A} = A$. Since $f(f^{-1}(B)) \subseteq B$, we know that

$$f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B} = B.$$

Applying f^{-1} gives

$$\overline{A} = f^{-1}(f(\overline{A})) \subseteq f^{-1}(B) = A.$$

It follows that A is closed. ■

Fri, Sept. 15

10.1. **Accumulation Points.** Ok, so we have learned that points in $\overline{A}$ are good enough to determine continuity of functions, but these points are not necessarily limits of sequences in A. It turns out that there is an alternative characterization of these points.

Definition 10.7. Let X be a space and $A \subseteq X$. A point $x \in X$ is said to be an **accumulation point** (or **cluster point** or **limit point**) of A if

every neighborhood of x contains a point of A other than x itself.

Let us write $\mathrm{acc}(A)$ for the set of accumulation points of A.

Example 10.8.
(1) Let $A = (0, 1) \subseteq \mathbb{R}$. Then $\mathrm{acc}(A) = [0, 1]$.
(2) Let $A = \{0, 1\} \subseteq \mathbb{R}$. Then $\mathrm{acc}(A) = \emptyset$.
(3) Let $A = [0, 1) \cup \{2\}$. Then $\mathrm{acc}(A) = [0, 1]$.
(4) Let $A = \{1/n\} \subseteq \mathbb{R}$. Then $\mathrm{acc}(A) = \{0\}$.

The following result follows immediately from our neighborhood characterization of the closure of a set.

Proposition 10.9. *A point x is an acc. point of A if and only if $x \in \overline{A \setminus \{x\}}$.*

Certainly $A \setminus \{x\} \subseteq A$, and the closure operation preserves containment, so it follows that $\mathrm{acc}(A) \subseteq \overline{A}$. From the previous examples, we see that this need not be an equality. We also have $A \subseteq \overline{A}$, and it follows that

$$A \cup \mathrm{acc}(A) \subseteq \overline{A}.$$

Proposition 10.10. *For any subset $A \subseteq X$, we have*

$$A \cup \mathrm{acc}(A) = \overline{A}.$$

Proof. It remains to show that every point in the closure is either in A or in $\mathrm{acc}(A)$. Let $x \in \overline{A}$, but suppose that $x \notin A$. Then, by the neighborhood criterion, we have that for every neighborhood N of x, $N \cap A \neq \emptyset$. But since $x \notin A$, it follows that $N \cap (A \setminus \{x\}) \neq \emptyset$. In other words, $x \in \mathrm{acc}(A)$. ■

Note that, although the motivation came from looking at sequences, there is no direct relation between accumulation points of A and limits of sequences in A.

We already saw an example of a point in the closure which is not the limit of a sequence. On the other hand, we can ask

Question 10.11. *If (a_n) is a sequence in A and $a_n \to x$, is $x \in \operatorname{acc}(A)$?*

Answer. No. Take $A = \{x\}$ and $a_n = x$. But, if we require that $x \notin A$, then the answer is yes.

As the example $X = \mathbb{R}^n$ suggests, sequences and closed sets are much better behaved for metric spaces.

Proposition 10.12 (The sequence lemma)**.** *Let $A \subseteq X$ and suppose that X is a metric space. Then $x \in \overline{A}$ if and only if x is the limit of a sequence in A.*

Proof. Let $S = \{1/n\}_{n\in\mathbb{N}} \cup \{0\}$, given the subspace topology from $\mathbb{R}$. Then a convergent sequence in a topological space is precisely the same as a continuous map from S to that topological space. We will also write $S_{>0} = S \setminus \{0\}$.

Suppose $a_n \to x$. Then this sequence gives a continuous map $a : S \longrightarrow X$ such that $a(S_{>0}) \subseteq A$. By Proposition 10.6, we know that

$$\operatorname{im}(a) = a(S) = a(\overline{S_{>0}}) \subseteq \overline{A}.$$

In particular, $x = a(0) \in \overline{A}$. This part of the argument does not require X to be metric.

On the other hand, suppose $x \in \overline{A}$. For each n, $B_{1/n}(x)$ is a neighborhood of x, and $x \in \overline{A}$, so $B_{1/n}(x) \cap A \neq \emptyset$. Let $a_n \in B_{1/n}(x) \cap A$. Then the sequence $a_n \to x$, and $a_n \in A$ by construction. ■

Note that by Example 10.3, it follows that the cocountable topology on $\mathbb{R}$ does not come from a metric on $\mathbb{R}$.

10.2. **Countability.** The last few lectures, we have seen that closed sets are not as easily understood in general as they are in the case of metric spaces. Although we will not want to restrict ourselves to metric spaces, it will nevertheless be helpful to have some good characterizations of the "reasonable" spaces. We mention here a few of these properties.

One property of metric spaces that we used recently was the existence of the balls of radius $1/n$.

Definition 10.13. A space X is **first-countable** if, for each $x \in X$, there is a countable collection $\{U_n\}$ of neighborhoods of x such that any other neighborhood contains at least one of the U_n.

This was the key property used in proving that, in a metric space, an accumulation point of $A \subseteq X$ is the limit of an A-sequence. Thus, we have

Proposition 10.14. *Let $f : X \longrightarrow Y$ be a function, where X is first-countable. Then f is continuous if and only if f takes convergent sequences in X to convergent sequences in Y.*

We will return to first-countable (and second-countable) spaces later in the course.

Example 10.15. Again, Proposition 10.14 implies that $X = \mathbb{R}_{\text{cocountable}}$ is not first-countable. We can see this directly as follows. Let $x \in X$ and suppose that $\{U_n\}$ is a collection of neighborhoods of x. By definition, each U_n is open and misses only countably many real numbers. Write $C_n = \mathbb{R} \setminus U_n$. Then $C = \bigcup_n C_n$ is also countable and is therefore a proper subset of $\mathbb{R}$. Let $z \neq x$ be some point in the complement of C. Then $C_2 = C \cup \{z\}$ is countable and strictly contains each C_n. Then $U_2 = X \setminus C_2$ is a neighborhood of x which is strictly contained in each U_n. Thus X is not first-countable.

Mon, Sept. 18

10.3. **Hausdorff Spaces.** Another important property of metric spaces is the Hausdorff property.

Definition 10.16. A space X is said to be Hausdorff (also called T_2) if, given any two points x and y in X, there are disjoint open sets U and V with $x \in U$ and $y \in V$.

This is a somewhat mild "separation property" that is held by many spaces in practice and that also has a number of nice consequences.

The Hausdorff property forces sequences to behave well, in the following sense.

Proposition 10.17. *In a Hausdorff space, a sequence cannot converge simultaneously to more than one point.*

Proof. Suppose $x_n \to x$ and $x_n \to y$. Every neighborhood of x contains a tail of x_n, as does any neighborhood of y. It follows that no neighborhood of x is disjoint from any neighborhood of y. Since X is Hausdorff, this forces $x = y$. ∎

Proposition 10.18. *Every metric space is Hausdorff.*

Proof. If $x \neq y$, let $d = d(x, y) > 0$. Then the balls of radius $d/2$ centered at x and y are the needed disjoint neighborhoods. ∎

However, of the (many, many) topologies on a finite set, the only one that is Hausdorff is the discrete topology. Indeed, if points are closed, then every subset is closed, as it is a finite union of points.

Here is one more nice consequence of this property.

Proposition 10.19. *If X is Hausdorff, then points are closed in X. (A space is called T_1 if points are closed.)*

Proof. The neighborhood crtierion for the complement $X \setminus \{x\}$ is easy to verify. ∎

11. Gluing Lemma

In Calculus, you saw functions defined piecewise, and one-sided limits were typically employed to establish continuity. There is an analogue of this type of construction for spaces.

Lemma 11.1 (Glueing/Pasting Lemma)**.** *Let $X = A \cup B$, where either (1) both A and B are open in X or (2) both A and B are closed in X. Then a function $f : X \longrightarrow Y$ is continuous if and only if the restrictions $f_{|A}$ and $f_{|B}$ are both continuous.*

Proof. ($\Rightarrow$) We already proved this in Proposition 8.4.

($\Leftarrow$) We give the proof assuming they are both open. Let $V \subseteq Y$ be open. We wish to show that $f^{-1}(V) \subseteq X$ is open. Let's restrict to A. We have $f^{-1}(V) \cap A = f_{|A}^{-1}(V)$. Since $f_{|A}$ is continuous, it follows that $f_{|A}^{-1}(V)$ is open (in A). Since A is open in X, it follows that $f_{|A}^{-1}(V)$ is also open in X. The same argument shows that $f^{-1}(V) \cap B$ is open in X. It follows that their unoin, which is $f^{-1}(V)$, is open in X. ∎

Example 11.2. For example, we can use this to paste together the continuous absolute value function $f(x) = |x|$, as a function $\mathbb{R} \longrightarrow \mathbb{R}$. We get this by pasting the continuous functions $\iota : [0, \infty) \longrightarrow \mathbb{R}$, $x \mapsto x$, and $(-\infty, 0] \cong [0, \infty) \longrightarrow \mathbb{R}$, $x \mapsto -x$.

Example 11.3. Let's look at an example of a discontinuous function, for example

$$f(x) = \begin{cases} 1 & x \neq 1 \\ 2 & x = 1. \end{cases}$$

We can get this by pasting together two constant functions, but the domains are $\mathbb{R} \setminus \{1\}$ and $\{1\}$, one of which is open but not closed, and the other of which is closed but not open.

Example 11.4. Let $X = [0,1] \cup [2,3]$, given the subspace topology from $\mathbb{R}$. Note that in this case each of the subsets $A = [0,1]$ and $B = [2,3]$ is **both** open and closed, so we can specify a continuous function on X by giving a pair of continuous functions, one on A and the other on B.

11.1. **Homeomorphisms.** Finally, we start to look at the idea of sameness. Two sets are thought of as the same if there is a bijection between them. A bijection is simply an invertible function. More generally, we have the following idea.

Definition 11.5. A "morphism" $f : X \longrightarrow Y$ is said to be an **isomorphism** if there is a $g : Y \longrightarrow X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Again, an isomorphism between sets is simply a bijection. In topology, this is called a **homeomorphism**. In other words, a homeomorphism is a continuous function with a continuous inverse. Since such a map is invertible, clearly it must be one-to-one and onto, but it is **not** true that every continuous bijection is a homeomorphism. Before we look at some examples, let's look at some non-examples.

Example 11.6.

(1) Any time a set is equipped with two topologies, one of which is a refinement of the other, the identity map is a continuous bijection (in one direction) that is not a homeomorphism. For instance, we have the following such examples

$$\mathrm{id} : \mathbb{R} \longrightarrow \mathbb{R}_{\text{cofinite}}, \qquad \mathrm{id} : \mathbb{R}_{\text{cocountable}} \longrightarrow \mathbb{R}_{\text{cofinite}} \qquad \mathrm{id} : \mathbb{R}_{\text{discrete}} \longrightarrow \mathbb{R}$$

(2) Consider the exponential map $\exp : [0,1) \longrightarrow S^1$ given by $\exp(x) = e^{2\pi i x}$. This is a continuous bijection, but it is not a homeomorphism. Since homeomorphisms have continuous inverses, they must take open sets to open sets and closed sets to closed sets. But we see that exp does not take the open set $U = [0, 1/2)$ to an open set in S^1. The point $\exp(0) = (1,0)$ has no neighborhood that is contained in $\exp(U)$.

Example 11.7.

(1) Consider $\tan : (0, \frac{\pi}{2}) \longrightarrow (0, \infty)$. This is a continuous bijection with continuous inverse (given by arctangent)

(2) Consider $\ln : (0,\infty) \longrightarrow \mathbb{R}$. This is a continuous bijection with inverse e^x. Composing homeomorphisms produces homeomorphisms, and we therefore get a homeomorphism

$$(0,1) \xrightarrow{\cong} (0, \frac{\pi}{2}) \xrightarrow{\cong} (0,\infty) \xrightarrow{\cong} \mathbb{R}.$$

(3) We similarly get a homeomorphism $\tan : [0, \frac{\pi}{2}) \xrightarrow{\cong} [0, \infty)$. It follows that we have

$$[0,1) \cong [0,\infty) \qquad \text{and} \qquad (0,1] \cong [0,\infty).$$

(4) One can similarly get $B^n_r(x) \cong \mathbb{R}^n$ for any n, r, and x.

The above example shows that there really are only three intervals, up to homeomorphism: the open interval, the half-open interval, and the closed interval.

We say that two spaces are **homeomorphic** if there is a homeomorphism between them (and write $X \cong Y$ as above). This is the notion of "sameness" for spaces. One of the major overarching questions for this course will be: how can we tell when two spaces are the same or are actually different?

A standard way to show that two spaces are not homeomorphic is to find a property that one has and the other does not. For instance every metric space is Hausdorff, so no non-Hausdorff space is the "same" as a metric space. But what property distinguishes the 3 interval types above? As we learn about more and more properties of spaces, this question will become easier to answer.

In the exponential example from last time, we noted that homeomorphisms must take open sets to open sets. Such a map is called an **open map**. Similarly, a **closed map** takes closed sets to closed sets.

Proposition 11.8. *Let $f : X \longrightarrow Y$ be a continuous bijection. The following are equivalent:*

(1) *f is a homeomorphism*
(2) *f is an open map*
(3) *f is a closed map*

If we drop the assumption that f is bijective, it is no longer true that being an open map is equivalent to being a closed map. For example, the inclusion $(0, 1) \longrightarrow \mathbb{R}$ is open but not closed, and the inclusion $[0, 1] \longrightarrow \mathbb{R}$ is closed but not open.

Fri, Sept. 22

Part 3. Constructions

12. Products

Put on your hard hats! We turn now to the construction phase. In section 3, we considered the product of metric spaces: let's define the product for topological spaces. We already know what property it should satisfy: we want it to be true that mapping continuously from some space Z into the product $X \times Y$ should be the same as mapping separately to X and to Y. Another way to describe this is that we want $X \times Y$ to be the "universal" example of a space with a pairs of maps to X and Y.

Well, if the projection $p_X : X \times Y \longrightarrow X$ is to be continuous, we need $p_X^{-1}(U) = U \times Y$ to be open whenever $U \subseteq X$ is open. Similarly, we need $X \times V$ to be open if $V \subseteq Y$ is open. We are forced to include these open sets, but we don't want to throw in anything extra that we don't need. In other words, we want the product topology on $X \times Y$ to be the coarsest topology containing the sets $U \times Y$ and $X \times V$.

Note that if we consider the collecion

$$\mathcal{B} = \{U \times Y \mid U \subseteq X \text{ open}\} \cup \{X \times V \mid V \subseteq Y \text{ open}\},$$

this cannot be a basis because it fails the intersection property. A typical intersection is

$$(U \times Y) \cap (X \times V) = U \times V,$$

and if we consider all sets of this form, we do get a basis.

Definition 12.1. Given spaces X and Y, the **product topology** on $X \times Y$ has basis given by sets of the form $U \times V$, where U and V are open in X and Y, respectively.

This satisfies the universal property of a product. We have engineered the definition to make this so, but we will check this anyway. First, we make a little detour.

We pointed out above that if we considered the collection

$$\mathcal{B} = \{U \times Y\} \cup \{X \times V\},$$

we would not have a basis, as the intersection property failed. We remedied this by considering instead intersections of elements of $\mathcal{B}$. This is a useful idea that shows up often.

Given a set X, a collection $\mathcal{C}$ of subsets of X is called a **prebasis** for a topology on X if the collection covers X. Actually, in all of the textbooks, this is called a subbasis, but that is a terrible name, since it suggests that it is a basis. I will try to stick with the better name of prebasis.

We can then get a basis from the prebasis by considering finite intersections of prebasis elements.

Example 12.2. The collection of rays (a, ∞) and $(-\infty, b)$ give a prebasis for the standard topology on $\mathbb{R}$.

We introduced the product topology above and mentioned the universal property, but let's spend a little bit of time with it to really nail down the concept.

Theorem-Definition 12.3. *Let X and Y be spaces. Then $X \times Y$, together with the projection maps*

$$p_X : X \times Y \longrightarrow X \quad \text{and} \quad p_Y : X \times Y \longrightarrow Y,$$

satisfies the following "universal property": given any space Z and maps $g : Z \longrightarrow X$ and $h : Z \longrightarrow Y$, there is a unique continuous map $f : Z \longrightarrow X \times Y$ such that

$$g = p_X \circ f, \qquad h = p_Y \circ f.$$

$$\begin{tikzcd}
& & X \\
Z \arrow[urr, bend left, "g"] \arrow[r, dashed, "\exists! f"] \arrow[drr, bend right, "h"'] & X \times Y \arrow[ur, "p_X"'] \arrow[dr, "p_Y"] & \\
& & Y
\end{tikzcd}$$

Proof. The uniqueness is clear: if there exists such a continuous map f, then the conditions force this to be $f = (g, h)$. The only question is whether or not f is continuous. Consider a typical basis element $U \times V$ for the product topology on $X \times Y$. Then

$$\begin{aligned} f^{-1}(U \times V) = \{z \in Z \mid f(z) \in U \times V\} &= \{z \in Z \mid g(z) \in U \text{ and } h(z) \in V\} \\ &= g^{-1}(U) \cap h^{-1}(V), \end{aligned}$$

which is an intersection of open sets and therefore open. ■

Ok, so we showed that $X \times Y$ satisfies this property, but why do we call this a "universal property"?

Proposition 12.4. *Suppose W is a space with continuous maps $q_X : W \longrightarrow X$ and $q_Y : W \longrightarrow Y$ also satisfying the property of the product. Then W is homeomorphic to $X \times Y$.*

Proof. The universal property for $X \times Y$ gives us a map $f : W \longrightarrow X \times Y$.

$$\begin{tikzcd}
& & X \\
W \arrow[urr, bend left, "q_X"] \arrow[r, dashed, "\exists! f"] \arrow[drr, bend right, "q_Y"'] & X \times Y \arrow[ur, "p_X"'] \arrow[dr, "p_Y"] & \\
& & Y
\end{tikzcd}$$

But W also has a universal property, so we get a map $\varphi : X \times Y \longrightarrow W$ as well.

$$\begin{tikzcd}
& & X \\
X \times Y \arrow[urr, bend left, "p_X"] \arrow[r, dashed, "\exists! \varphi"] \arrow[drr, bend right, "p_Y"'] & W \arrow[ur, "q_X"'] \arrow[dr, "q_Y"] & \\
& & Y
\end{tikzcd}$$

Now make Pacman eat Pacman!

$$\begin{tikzcd}
& & & X \\
W \arrow[urrr, bend left, "q_X"] \arrow[drrr, bend right, "q_Y"'] \arrow[r, dotted, "f"] & X \times Y \arrow[urr, dotted, "p_X"] \arrow[drr, dotted, "p_Y"'] \arrow[r, dotted, "\varphi"] & W \arrow[ur, "q_X"'] \arrow[dr, "q_Y"] & \\
& & & Y
\end{tikzcd}$$

We have a big diagram, but if we ignore all dotted lines, there is an obvious horizontal map $W \longrightarrow W$ to fill in the diagram, namely the id_W. Since the universal property guarantees that there is a **unique** way to fill it in, we find that $\varphi \circ f = \mathrm{id}_W$. Reversing the pacmen gives the other equality $f \circ \varphi = \mathrm{id}_{X \times Y}$. In other words, f is a homeomorphism, and $\varphi = f^{-1}$. ∎

> This argument may seem strange the first time you see it, but it is a typical argument that applies any time you define an object via a universal property. The argument shows that any two objects satisfying the universal property must be "the same".

Proposition 12.5. *Let $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ be continuous. Then the product map $f \times f' : X \times X' \longrightarrow Y \times Y'$ is also continuous.*

Proof. This follows very easily from the universal property. If we want to map continuously to $Y \times Y'$, it suffices to specify continuous maps to Y and Y'. The continuous map $X \times X' \longrightarrow Y$ is the composition

$$X \times X' \xrightarrow{p_X} X \xrightarrow{f} Y,$$

and the other needed map is the composition

$$X \times X' \xrightarrow{p_{X'}} X' \xrightarrow{f'} Y'.$$

∎

Ok, so we understand $X \times Y$ as a topological space. What about a product of more than two spaces? Well, if we have a finite collection $X_1, \ldots, X_n$ of spaces, the product topology on $X_1 \times \cdots \times X_n$ has basis given by the $U_1 \times \cdots \times U_n$, or equivalently, prebasis given by the $p_j^{-1}(U_j)$. Note that this is equivalent because the basis element $U_1 \times \cdots \times U_n$, is a finite intersection of the prebasis elements $p_j^{-1}(U_j)$.

But what about the product of an *arbitrary* number of spaces? Here, the property we want is that whenever we have a space Z and maps $f_j : Z \longrightarrow X_j$ for all i, then there should be a unique continuous map $f : Z \longrightarrow \prod_{j \in J} X_j$ such that $p_j \circ f = f_j$.

Just as for finite products, we want the projection maps $p_j : \prod_{j \in J} \longrightarrow X_j$ to be continuous. This forces each $p_j^{-1}(U_j)$ to be continuous, and we can again choose these for a prebasis. We thus get a basis consisting of finite intersections $p_{j_1}^{-1}(U_{j1}) \cap \cdots \cap p_{j_k}^{-1}(U_{j_k})$.

Definition 12.6. Given spaces X_j, one for each $j \in J$, the **product topology** on $\prod_{j \in J} X_j$ has basis consisting of the $p_{j_1}^{-1}(U_{j1}) \cap \cdots \cap p_{j_k}^{-1}(U_{j_k})$.

Mon, Sept. 25

Last time, we introduced the *product topology* on $\prod_{j \in J} X_\alpha$, which had basis

$$\mathcal{B}_{\text{prod}} = \left\{ \prod_j U_j \mid U_j \subseteq X_j \text{ is open, and only finitely many } U_j \text{ are } \textbf{proper} \text{ subsets} \right\}.$$

Proposition 12.7. *The product topology on* $\prod_{j \in J} X_j$*, as defined above, satisfies the following universal property: given any space* Z *and continuous maps* $f_j : Z \longrightarrow X_j$ *for all* $j \in J$*, there is a unique continuous* $f : Z \longrightarrow \prod_{j \in J} X_j$ *such that* $p_j \circ f = f_j$ *for all* $j \in J$*.*

Proof. The same proof as that given in 12.3 works here. Given the maps f_j, we define f by $f(z)_j = f_j(z)$. Again, the equations $p_j \circ f = f_j$ force this choice on us. The only question is whether this makes f into a continuous map. Since the topology on $\prod_{j \in J} X_j$ is defined by the prebasis elements $p_j^{-1}(U_j)$, it suffices to show that each of these pulls back to an open set. But

$$f^{-1}(p_j^{-1}(U_j)) = (p_j \circ f)^{-1}(U_j) = f_j^{-1}(U_j),$$

which is open since f_j is continuous. ■

12.1. **Box Topology.** We have defined the *product topology* on $\prod_{j \in J} X_j$. But there is another obvious guess, coming from the answer for finite products. We can think about the basis consisting of products $\prod_j U_j$. This is no longer equivalent to the product topology!

Definition 12.8. Suppose given a collection of spaces X_j. The **box topology** on $\prod_{j \in J} X_j$ is generated by the basis $\left\{ \prod_{j \in J} U_j \right\}$.

As discussed above, the box topology has more open sets; in other words, the box topology is finer than the product topology. To see that the box topology does not have the universal property we want, consider the following example: let $\Delta : \mathbb{R} \longrightarrow \prod_{n \in \mathbb{N}} \mathbb{R}$ be the diagonal map, all of whose component maps are simply the identity. For each n, let $I_n = (\frac{-1}{n}, \frac{1}{n})$. In the box topology, the subset $I = \prod_n I_n \subseteq \prod_n \mathbb{R}$ is an open set, but

$$\Delta^{-1}(I) = \bigcap_n \text{id}^{-1}(I_n) = \bigcap_n I_n = \{0\}$$

is not open. So the diagonal map is not continuous in the box topology!

Since we are now considering arbitrary products, it may be useful to stop and clarify what we mean. For instance, we might want to consider a countable infinite product $\mathbb{R} \times \mathbb{R} \times \dots$.

Let X_j, for $j \in J$, be sets. The cartesian product $\prod_{j \in J} X_j$ is the collection of tuples (x_j), where $x_j \in X_j$. This means that for each $j \in J$, we want an element $x_j \in X_j$. In other words, we should have a function

$$x_{(-)} : J \longrightarrow X = \bigcup_j X_j$$

with the condition that this function satisfies $x_j \in X_j$. With this language, the "projection" $\prod_{j \in J} X_j \longrightarrow X_j$ is simply the restriction along $\{j\} \hookrightarrow J$.

In the case that all X_j are the same set X, then $\prod_{j \in J} X_j$ is simply the set of functions $J \longrightarrow X$. So, the countably infinite product of $\mathbb{R}$ with itself is synonymous with the collection of sequences in $\mathbb{R}$.

Example 12.9. We mentioned above that the set of sequences in $\mathbb{R}$ is the infinite product $\prod_n \mathbb{R}$. What does a neighborhood of a sequence (x_n) look like in the product topology? We are only allowed to constrain finitely many coordinates, so a neighborhood consists of all sequences that are near to (x_n) in some fixed, finitely many coordinates.

Wed, Sept. 27

Proposition 12.10. *Let $A_j \subseteq X_j$ for all $j \in J$. Then*

$$\prod_j \overline{A_j} = \overline{\prod_j A_j}$$

in both the product and box topologies.

Proof. As usual, we have two subsets of $\prod_j X_j$ we want to show are the same, so we establish that each is a subset of the other. The following proof works in both topologies under consideration.

($\subseteq$) Let $(x_j) \in \prod \overline{A_j}$. We use the neighborhood criterion of the closure to show that $(x_j) \in \overline{\prod_j A_j}$. Thus let $U = \prod_j U_j$ be a basic open neighborhood of (x_j). Then for each j, U_j is a neighborhood of x_j. Since $x_j \in \overline{A_j}$, it follows that U_j must meet A_j in some point, say y_j. It then follows that $(y_j) \in U \cap \prod_j A_j$. By the neighborhood criterion, it follows that $(x_j) \in \overline{\prod_j A_j}$.

($\supseteq$) For the other direction, we simply use that the projection is continuous:

$$p_j\left(\overline{\prod_j A_j}\right) \subseteq \overline{p_j\left(\prod_j A_j\right)} = \overline{A_j}.$$

This shows that

$$\overline{\prod_j A_j} \subseteq \prod_j \overline{A_j}.$$

■

Note that this implies that an (arbitrary) product of closed sets is closed, using either the product or box topologies. In particular, I^2 is closed in $\mathbb{R}^2$ and T^2 is closed in $\mathbb{R}^4$.

Proposition 12.11. *Suppose X_j is Hausdorff for each $j \in J$. Then so is $\prod_j X_j$ in both product and box topologies.*

Proof. Let $(x_j) \neq (x'_j) \in \prod_j X_j$. Then $x_\ell \neq x'_\ell$ for some particular ℓ. Since X_ℓ is Hausdorff, we can find disjoint neighborhoods U and U' of x_ℓ and x'_ℓ in X_ℓ. Then $p_\ell^{-1}(U)$ and $p_\ell^{-1}(U')$ are disjoint neighborhoods of (x_j) and (x'_j) in the product topology, so $\prod_j X_j$ is Hausdorff in the product topology.

For the box topology, we can either say that the above works just as well for the box topology, or we can say that since the box topology is a refinement of the product topology and the product topology is Hausdorff, it follows that the box topology must also be Hausdorff. ∎

The converse is true as well, assuming that each X_j is nonempty. To see this, we use the fact that a subspace of a Hausdorff space is Hausdorff. How do we view X_ℓ as a subspace of $\prod_j X_j$? We can think about an axis inclusion. Thus pick $y_j \in X_j$ for $j \neq \ell$. We define

$$a_\ell : X_\ell \longrightarrow \prod_j X_j$$

by

$$a_\ell(x)_j = \begin{cases} x & j = \ell \\ y_j & j \neq \ell. \end{cases}$$

Note that, by the universal property of the product, in order to check that a_ℓ is continuous, it suffices to check that each coordinate map is continuous. But the coordinate maps are the identity and a lot of constant maps, all of which are certainly continuous. The map a_ℓ is certainly injective (assuming all X_j are nonempty!), and it is an example of an embedding.

Definition 12.12. A map $f : X \longrightarrow Y$ is said to be an **embedding** if it is a homeomorphism onto its image $f(X)$, equipped with the subspace topology.

We already discussed injectivity and continuity of the axis inclusion a_ℓ, so it only remains to show this is open, as a map to $a_\ell(X_\ell)$. Let $U \subseteq X_\ell$ be open. Then

$$a_\ell(U) = p_\ell^{-1}(U) \cap a_\ell(X_\ell),$$

so $a_\ell(U)$ is open in the subspace topology on $a_\ell(X_\ell)$.

We will often do the above sort of exercise: if we introduce a new property or construction, we will ask how well this interacts with other constructions/properties.

Here is another example of an embedding.

Example 12.13. Let $f : X \longrightarrow Y$ be continuous and define the graph of f to be

$$\Gamma(f) = \{(x, y) \mid y = f(x)\} \subseteq X \times Y.$$

The function

$$\gamma : X \longrightarrow X \times Y, \qquad \gamma(x) = (x, f(x))$$

is an embedding with image $\Gamma(f)$.

Let us verify that this is indeed an embedding. Injectivity is easy (this follows from the fact that one of the coordinate maps is injective), and continuity comes from the universal property for the product $X \times Y$ since id_X and f are both continuous. Note that $(p_X)_{|\Gamma(f)}$, which is continuous since it is the restriction of the continuous projection p_X, provides an inverse to γ.

Fri, Sept. 29

13. Coproduct

What happens if we turn all of the arrows around in the defining property of a product? We might call such a thing a "coproduct". To be precise we would want a space that is universal among spaces equipped with maps *from* X and Y. In other words, given a space Z and maps $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$, we would want a unique map from the coproduct to Z, making the following diagram commute.

$$\begin{tikzcd}
X \arrow[dr] \arrow[drr, bend left, "f"] & & \\
& X \amalg Y \arrow[r, dashed, "\exists! h"] & Z \\
Y \arrow[ur] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

The glueing lemma gave us exactly such a description, in the case that our domain space X was made up of *disjoint* open subsets A and B. In general, the answer here is given by the **disjoint union**.

Recall that, as a set, the disjoint union of sets X and Y is the subset

$$X \amalg Y \subseteq (X \cup Y) \times \{1, 2\},$$

where $X \amalg Y = (X \times \{1\}) \cup (Y \times \{2\})$. More generally, given sets X_j for $j \in J$, their disjoint union $\coprod_j X_j$ is the subset

$$\coprod_j X_j \subseteq \left(\bigcup_j X_j \right) \times J$$

given by

$$\coprod_j X_j = \bigcup_j (X_j \times \{j\}) .$$

There are natural inclusions $\iota_X : X \longrightarrow X \amalg Y$ or more generally $\iota_{X_j} : X_j \hookrightarrow \coprod_j X_j$. We topologize the coproduct by giving it the finest topology such that all ι_{X_j} are continuous. In other words, a subset $U \subseteq \coprod_j X_j$ is open if and only if $\iota_j^{-1}(U) \subseteq X_j$ is open for all j.

Note that in the case of a coproduct of two spaces, the subspace topology on $X \subseteq X \amalg Y$ agrees with the original topology on X. Furthermore, both X and Y are open in $X \amalg Y$, so the universal property for the coproduct is precisely the glueing lemma.

On Friday, we introduced the idea of a coproduct, which is dual to the product. In the case of a space X which happens to be the union of two open, disjoint, subspaces A and B, then the glueing lemma told us that X satisfies the correct property to be the coproduct $X = A \amalg B$.

For a more general coproduct $\coprod_j X_j$, we declared $U \subseteq \coprod_j X_j$ to be open if and only if $\iota_j^{-1}(U)$ is open for all j. Let's verify that this satisfies the universal property.

Thus let $f_j : X_j \longrightarrow Z$ be continuous for all $j \in J$. It is clear that, set-theoretically, the various images $\iota_j(X_j)$ inside the coproduct are disjoint and that their union is the entire coproduct. So to define a function on the coproduct, it suffices to define a function on each $\iota_j(X_j)$. But each ι_j is injective, in other words a bijection onto its image, so defining $f_{|\iota_j(X_j)}$ is equivalent to defining

$f_{|\iota_j(X_j)} \circ \iota_j$. But the latter, according to the universal property, is supposed to be f_j. So the upshot of all of this is that there is no choice in how we define the function f. As usual, we only need verify that this function f is continuous.

Let $V \subseteq Z$ be open. We wish to know that $f^{-1}(V)$ is open in $\coprod_j X_j$. But according to the topology on the coproduct, this amounts to showing that each $\iota_j^{-1} f^{-1}(V)$ is open. But this is $(f \circ \iota_j)^{-1}(V) = f_j^{-1}(V)$, which is open by the assumption that each f_j is continuous.

Example 13.1.
1. Consider $X = [0, 1]$ and $Y = [2, 3]$. Then in this case $X \amalg Y$ is homeomorphic to the subspace $X \cup Y$ of $\mathbb{R}$. The same is true of these two intervals are changed to be open or half-open.
2. Consider $X = (0, 1)$ and $Y = \{1\}$. Then $X \amalg Y$ is **not** homeomorphic to $(0, 1) \cup \{1\} = (0, 1]$. The singleton $\{1\}$ is open in $X \amalg Y$ but not in $(0, 1]$. Instead, $X \amalg Y$ is homeomorphic to $(0, 1) \cup \{2\}$.
3. Similarly $(0, 1) \amalg [1, 2]$ is homeomorphic to $(0, 1) \cup [2, 3]$ but not to $(0, 1) \cup [1, 2] = (0, 2]$.
4. In yet another similar example, $(0, 2) \amalg (1, 3)$ is homeomorphic to $(0, 1) \cup (2, 3)$ but not to $(0, 2) \cup (1, 3) = (0, 3)$.

Proposition 13.2. *Let X_i be spaces, for $i \in I$. Then $\coprod_i X_i$ is Hausdorff if and only if all X_i are Hausdorff.*

Mon, Oct. 2

Proof. This is even easier than for products. First, X_i always embeds as a subspace of the coproduct, so it follows that X_i is Hausdorff if the coproduct is as well. On the other hand, suppose all X_i are Hausdorff and suppose that $x \neq y$ are points of $\coprod_i X_i$. Either x and y come from different X_i's, in which case the X_i's themselves serve as the disjoint neighborhoods. The alternative is that x and y live in the same Hausdorff X_i, but then we can find disjoint neighborhoods in X_i. ■

14. Quotients

The next important construction is that of a quotient, or identification space.

The general setup is that we have a surjective map $q : X \longrightarrow Y$, which we view as making an identification of points in X. More precisely, suppose that we have an equivalence relation $\sim$ on X. We can consider the set $X/\sim$ of equivalence classes in X. There is a natural surjective map $q : X \longrightarrow X/\sim$ which takes $x \in X$ to its equivalence class.

And in fact every surjective map is of this form. Suppose that $q : X \longrightarrow Y$ is surjective. We define a relation on X by saying that $x \sim x'$ if and only if $q(x) = q(x')$. Then the function $X/\sim \longrightarrow Y$ sending the class of x to $q(x)$ is a bijection.

We want to mimic the above situation in topology, but to understand what this should mean, we first look at the universal property of the quotient for sets. This says: if $f : X \longrightarrow Z$ is a function that is constant on the equivalence classes in X, then there is a (unique) factorization $g : X/\sim \longrightarrow Z$ with $g \circ q = f$.

We want to have a similar setup in topology. Said in the equivalence relation framework, given a space X and a relation $\sim$ on X, we want a continuous map $q : X \longrightarrow Y$ such that given any space Z with a continuous map $f : X \longrightarrow Z$ which is constant on equivalence classes, there is a unique continuous map $g : Y \longrightarrow Z$ such that $g \circ q = f$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q"'] & & Z \\ & Y \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

By considering the cases in which Z is a set with the trivial topology, so that maps to Z are automatically continuous, we can see that on the level of sets $q : X \longrightarrow Y$ must be $X \longrightarrow X/\sim$. It remains only to specify the topology on $Y = X/\sim$.

We want the topological quotient to be the universal example of a continuous map out of X which is constant on equivalence classes. Since we want to construct maps *out of* Y, this suggests we should include as many open sets as possible in Y. This leads to the following definition.

Definition 14.1. We say that a surjective map $q : X \longrightarrow Y$ is a **quotient map** if $V \subseteq Y$ is open if and only if $q^{-1}(V)$ is open in X.

One implication is the definition of continuity, but the other is given by our desire to include as many opens as we can.

Proposition 14.2. *(Universal property of the quotient) Let $q : X \longrightarrow Y$ be a quotient map. If Z is any space, and $f : X \longrightarrow Z$ is any continuous map that is constant on the fibers*[2] *of q, then there exists a unigue continuous $g : Y \longrightarrow Z$ such that $g \circ q = f$.*

[2] A "fiber" is simply the preimage of a point.

Proof. It is clear how g must be defined: $g(y) = f(x)$ for any $x \in q^{-1}(y)$. It remains to show that g is continuous. Let $W \subseteq Z$ be open. We want $g^{-1}(W) \subseteq Y$ to be open as well. By the definition of a quotient map, $g^{-1}(W)$ is open if and only if $q^{-1}(g^{-1}(W)) = (g \circ q)^{-1}(W) = f^{-1}(W)$ is open, so we are done by continuity of f. ■

Example 14.3. Define $q : \mathbb{R} \longrightarrow \{-1, 0, 1\}$ by

$$q(x) = \begin{cases} 0 & x = 0 \\ \frac{|x|}{x} & x \neq 0. \end{cases}$$

What is the resulting topology on $\{-1, 0, 1\}$? The points -1 and 1 are open, and the only open set containing 0 is the whole space.

Note that this example shows that a quotient of a Hausdorff space need not be Hausdorff.

Proposition 14.4. *Let $q : X \longrightarrow Y$ be a continuous, surjective, open map. Then q is a quotient map. The same is true if q is closed instead of open.*

Proof. One implication is simply the definition of continuity. For the other, suppose that $V \subseteq Y$ is a subset such that $q^{-1}(V) \subseteq X$ is open. Then $q(q^{-1}(V))$ is open since q is open. Finally, we have $V = q(q^{-1}(V))$ since q is surjective. ■

The converse is not true, however, as the next example shows.

Example 14.5. Consider $q; \mathbb{R} \longrightarrow [0, \infty)$ given by

$$q(x) = \begin{cases} 0 & x \leq 0 \\ x & x \geq 0. \end{cases}$$

The quotient topology on $[0, \infty)$ is the same as the subspace topology it gets from $\mathbb{R}$. But this is not an open map, since the image of $(-2, -1)$ is $\{0\}$, which is not open.

14.1. **Saturated Open Sets.** We discussed last time the fact that a quotient map need not be open. Nevertheless, there is a class of open sets that are always carried to open sets.

Definition 14.6. Let $q : X \longrightarrow Y$ be a continuous surjection. We say a subset $A \subseteq X$ is **saturated** (with respect to q) if it is of the form $q^{-1}(V)$ for some subset $V \subseteq Y$.

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It follows that A is saturated if and only if $q^{-1}(q(A)) = A$. Recall that a **fiber** of a map $q : X \longrightarrow Y$ is the preimage of a single point. Then another description is that A is saturated if and only if it contains all fibers that it meets.

Proposition 14.7. *A continuous surjection $q : X \twoheadrightarrow Y$ is a quotient map if and only if it takes saturated open sets to saturated open sets.*

Proof. Exercise. ■

14.2. **Examples.**

Example 14.8. (Collapsing a subspace) Let $A \subseteq X$ be a subspace. We define a relation on X as follows: $x \sim y$ if both are points in A or if neither is in A and $x = y$. Here, we have one equivalence class for the subset A, and every point outside of A is its own equivalence class. Standard notation for the set $X/\sim$ of equivalence classes under this relation is X/A. The universal property can be summed up as saying that any map on X which is constant on A factors through the quotient X/A.

For example, we considered last time the example $\mathbb{R}/(-\infty, 0] \cong [0, \infty)$.

Example 14.9. Consider $\partial I \subseteq I$. The exponential map $e : I \longrightarrow S^1$ is constant on ∂I, so we get an induced continuous map $\varphi : I/\partial I \longrightarrow S^1$, which is easily seen to be a bijection. In fact, it is a homeomorphism. Once we learn about compactness, it will be easy to see that this is a closed map.

We show instead that it is open. A basis for $I/\partial I$ is given by $q(a,b)$ with $0 < a < b < 1$ and by $q([0,a) \cup (b,1])$ with again $0 < a < b < 1$. Since both are taken to basis elements for the subspace topology on S^1, it follows that φ is a homeomorphism.

Example 14.10. Generalizing the previous example, for any closed ball $D^n \subseteq \mathbb{R}^{n+1}$, we can consider the quotient $D^n/\partial D^n$. Exercise: define a surjective continuous map

$$q : D^n \longrightarrow S^n$$

taking the origin to the south pole and the boundary to the north pole. This then defines a continuous bijection $D^n/\partial D^n \longrightarrow S^n$, and we will see later in the course that this is automatically a homeomorphism.

$\boxed{\textbf{Fri, Oct. 6}}$

Exam day

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(**Examples continued . . .**)

Example 14.11. (Cylinder) On $I \times I$, we impose the relation $(0, y) \sim (1, y)$. The resulting quotient space is a cylinder, which can be identified with $S^1 \times I$.

Example 14.12. (Möbius band) On $I \times I$, we impose the relation $(0, y) \sim (1, 1-y)$. The resulting quotient space is the Möbius band M.

Example 14.13. (Torus) On $I \times I$, we impose the relation $(0, y) \sim (1, y)$ and also the relation $(x, 0) \sim (x, 1)$. The resulting quotient space is the torus $T^2 = S^1 \times S^1$. We recognize this as the product of two copies of example 14.8, but beware that in general a product of quotient maps need not be a quotient map.

Example 14.14. (Real projective space) On S^n we impose the equivalence relation $\mathbf{x} \sim -\mathbf{x}$. The resulting quotient space is known as n-dimensional real projective space and is denoted $\mathbb{RP}^n$.

Consider the case $n = 1$. We have the hemisphere inclusion $I \hookrightarrow S^1$ given by $x \mapsto e^{ix\pi}$. Then the composition $I \hookrightarrow S^1 \twoheadrightarrow \mathbb{RP}^1$ is a quotient map that simply identifies the boundary ∂I to a point. In other words, this is example 14.8 from above, and we conclude that $\mathbb{RP}^1 \cong S^1$. However, the higher-dimensional versions of these spaces are certainly not homeomorphic. We will return to this soon in Example 15.5.

Example 14.15. (Complex projective space) Consider S^{2n-1} as a subspace of $\mathbb{C}^n$. We then have the coordinate-wise multiplication by elements of $S^1 \cong U(1)$ on $\mathbb{C}^n$. This multiplication restricts to a multiplication on the subspace S^{2n-1}, and we impose an equivalence relation $(z_1, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n)$ for all $\lambda \in S^1$. The resulting quotient space is the complex projective space $\mathbb{CP}^n$.

15. Topological Groups

A number of the examples above have secretly been examples of a more general construction, namely the quotient under the action of a group.

Definition 15.1. A **topological group** is a based space (G, e) with a continuous multiplication $m : G \times G \longrightarrow G$ and inverse $i : G \longrightarrow G$ satisfying all of the usual axioms for a group.

Remark 15.2. Munkres requires all topological groups to satisfy the condition that points are closed. We will not make this restriction, though the examples we will consider will all satisfy this.

Example 15.3.
1. Any group G can be considered as a topological group equipped with the discrete topology. For instance, we have the cyclic groups $\mathbb{Z}$ and $C_n = \mathbb{Z}/n\mathbb{Z}$.
2. The real line $\mathbb{R}$ is a group under addition, This is a topological group because addition and multiplication by -1 are both continuous. Note that here $\mathbb{Z}$ is at the same time both a subspace and a subgroup. It is thus a topological subgroup.
3. If we remove zero, we get the multiplicative group $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ of real numbers.
   $\boxed{\textbf{Wed, Oct. 11}}$
4. Inside $\mathbb{R}^\times$, we have the subgroup $\{1, -1\}$ of order two.
5. $\mathbb{R}^n$ is also a topological group under addition. In the case $n = 2$, we often think of this as $\mathbb{C}$.
6. Again removing zero, we get the multiplicative group $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ of complex numbers.
7. Inside $\mathbb{C}^\times$ we have the subgroup of complex numbers of norm 1, aka the circle group $S^1 \cong U(1) = SO(2)$.

(8) This last example suggests that matrix groups in general are good candidates. For instance, we have the topological group $Gl_n(\mathbb{R})$. This is a subspace of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. The determinant mapping $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is polynomial in the coefficients and therefore continuous. The general linear group is the complement of $\det^{-1}(0)$. It follows that $Gl_n(\mathbb{R})$ is an open subspace of $\mathbb{R}^{n^2}$.

(9) Inside $Gl_n(\mathbb{R})$, we have the closed subgroups $Sl_n(\mathbb{R})$, $O(n)$, $SO(n)$.

Let G be a topological group and fix some $h \in G$. Define $L_h : G \longrightarrow G$ by $L_h(g) = hg$. This is left multiplication by h. The definition of topological group implies that this is continuous, as L_h is the composition

$$G \xrightarrow{(h,\mathrm{id})} G \times G \xrightarrow{m} G.$$

Moreover, $L_{h^{-1}}$ is clearly inverse to L_h and continuous by the same argument, so we conclude that each L_h is a homeomorphism. Since $L_h(e) = h$, we conclude that neighborhoods around h look like neighborhoods around e. Since h was arbitrary, we conclude that neighborhoods around one point look like neighborhoods around any other point. This implies that a space like the unoin of the coordinate axes in $\mathbb{R}^2$ cannot be given the structure of topological group, as neighborhoods around the origin do not resemble neighborhoods around other points.

15.1. **Group actions & orbit spaces.** The main reason for studying topological groups is to consider their *actions* on spaces.

Definition 15.4. Let G be a topological group and X a space. A **left action** of G on X is a map $a : G \times X \longrightarrow X$ which is associative and unital. This means that $a(g, a(h, x)) = a(gh, x)$ and $a(e, x) = x$. Diagrammatically, this is encoded as the following commutative diagrams

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mathrm{id}\times a"] \arrow[d, "m\times\mathrm{id}"'] & G \times X \arrow[d, "a"] \\ G \times X \arrow[r, "a"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "{e,\mathrm{id}}"] \arrow[dr, "\mathrm{id}"'] & G \times X \arrow[d, "a"] \\ & X. \end{tikzcd}$$

It is common to write $g \cdot x$ or simply gx rather than $a(g, x)$.

There is a similar notion of right action of G on X, given by a map $X \times G \longrightarrow X$ satisfying the appropriate properties.

Fri, Oct. 13

Given an action of G on a space X, we define a relation on X by $x \sim y$ if $y = g \cdot x$ for some g. The equivalence classes are known as **orbits** of G in X, and the quotient of X by this relation is typically written as X/G. Really, the notation X/G should be reserved for the quotient by a *right action* of G on X, and the quotient by a left action should be $G\backslash X$.

Example 15.5. (1) For any G, left multiplication gives an action of G on itself! This is a transitive action, meaning that there is only one orbit, and the quotient $G\backslash G$ is just a point.

Note that we saw above that, for each $h \in G$, the map $L_h : G \longrightarrow G$ is a homeomorphism. This generalizes to any action. For each $g \in G$, the map $a(g, -) : X \longrightarrow X$ is a homeomorphism.

(2) For any (topological) subgroup $H \leq G$, left multiplication by elements of H gives a left action of H on G. Note that an orbit here is precisely a right coset Hg. The quotient is $H\backslash G$, the set of right cosets of H in G.

(3) Consider the subgroup $\mathbb{Z} \leq \mathbb{R}$. Since $\mathbb{R}$ is abelian, we don't need to worry about about left vs. right actions or left vs. right cosets. We then have the quotient $\mathbb{R}/\mathbb{Z}$, which is again a topological group (again, $\mathbb{R}$ is abelian, so $\mathbb{Z}$ is normal).

What is this group? Once again, consider the exponential map $\exp : \mathbb{R} \longrightarrow S^1$ given by $\exp(x) = e^{2\pi i x}$. This is surjective, and it is a homomorphism since

$$\exp(x + y) = \exp(x)\exp(y).$$

The First Isomorphism Theorem in group theory tells us that $S^1 \cong \mathbb{R}/\ker(\exp)$, at least as a group. The kernel is precisely $\mathbb{Z} \leq \mathbb{R}$, and it follows that $S^1 \cong \mathbb{R}/\mathbb{Z}$ as a group. To see that this is also a homeomorphism, we need to know that $\exp : \mathbb{R} \longrightarrow S^1$ is a quotient map, but this follows from our earlier verification that $I \longrightarrow S^1$ is a quotient. Another way to think about this is that the universal property of the quotient gives us continuous maps $I/\partial I \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow I/\partial I$ which are inverse to each other.

(4) Similarly, we can think of $\mathbb{Z}^n$ acting on $\mathbb{R}^n$, and the quotient is $\mathbb{R}^n/\mathbb{Z}^n \cong (S^1)^n = T^n$.

(5) The group $Gl(n)$ acts on $\mathbb{R}^n$ (just multiply a matrix with a vector), but this is not terribly interesting, as there are only two orbits: the origin is a closed orbit, and the complement is an open orbit. Thus the quotient space consists of a closed point and an open point.

(6) More interesting is the action of the subgroup $O(n)$ on $\mathbb{R}^n$. Using the fact that orthogonal matrices preserve norms, it is not difficult to see that the orbits are precisely the spheres around the origin. We claim that the quotient is the space $[0, \infty)$ (thought of as a subspace of $\mathbb{R}$).

To see this, consider the continuous surjection $|-| : \mathbb{R}^n \longrightarrow [0, \infty)$. By considering how this acts on open balls, you can show that this is an open map and therefore a quotient. But the fibers of this map are precisely the spheres, so it follows that this is the quotient induced by the above action of $O(n)$.

(7) Let $\mathbb{R}^\times$ act on $\mathbb{R}^n$ via scalar multiplication. This action preserves lines, and within each line there are two orbits, one of which is the origin. Note that the only saturated open set containing 0 is $\mathbb{R}^n$, so the only neighborhood of 0 in the quotient is the entire space.

(8) Switching from n to $n+1$ for convenience, we can remove that troublesome 0 and let $\mathbb{R}^\times$ act on $X_{n+1} = \mathbb{R}^{n+1} \setminus \{0\}$. Here the orbits are precisely the lines (with origin removed). The quotient is $\mathbb{RP}^n$.

To see this, recall that we defined $\mathbb{RP}^n$ as the quotient of S^n by the relation $\mathbf{x} \sim -\mathbf{x}$. This is precisely the relation that arises from the action of the subgroup $C_2 = \{1, -1\} \leq \mathbb{R}^\times$ on $S^n \subseteq \mathbb{R}^{n+1}$.

Now notice that the map $\mathbb{R}^{n+1} \setminus \{0\} \longrightarrow S^n \times \mathbb{R}_{>0}$ given by $\mathbf{x} \mapsto \left(\frac{\mathbf{x}}{\|\mathbf{x}\|}, \|\mathbf{x}\|\right)$ is a homeomorphism. Next, note that we have an isomorphism $\mathbb{R}^\times \cong C_2 \times \mathbb{R}^\times_{>0}$. Thus the quotient $(\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^\times$ can be viewed as the two step quotient $\Big((S^{n-1} \times \mathbb{R}_{>0})/\mathbb{R}^\times_{>0})\Big)/C_2$. But $(\mathbb{R}^{n-1} \times \mathbb{R}_{>0})/\mathbb{R}^\times_{>0} \cong S^{n-1}$, so we are done.

We can think of $\mathbb{RP}^n$ in yet another way. Consider the following diagram:

$$\begin{tikzcd}
D^n \arrow[r] \arrow[d] & S^n \arrow[r] \arrow[d] & \mathbb{R}^{n+1} \setminus \{0\} \arrow[d] \\
D^n/\sim \arrow[r] & S^n/C_2 \arrow[r] & (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^\times
\end{tikzcd}$$

The map $D^n \longrightarrow S^n$ is the inclusion of a hemisphere. The relation on D^n is the relation $\mathbf{x} \sim -\mathbf{x}$, but only allowed *on the boundary* ∂D^n. All maps on the bottom are continuous bijections, and again we will see later that they are necessarily homeomorphisms.

Note that the relation we imposed on D^n does *not* come from an action of C_2 on D^n. Let us write $C_2 = \langle \sigma \rangle$. We can try defining

$$\sigma \cdot \mathbf{x} = \begin{cases} \mathbf{x} & \mathbf{x} \in \mathrm{Int}(D^n) \\ -\mathbf{x} & \mathbf{x} \in \partial(D^n), \end{cases}$$

where here the interior and boundary are taken in S^n. But this is not continuous, as the convergent sequence

$$\left(\sqrt{1 - \frac{1}{n}}, 0, \ldots, 0, \sqrt{\frac{1}{n}} \right) \to (1, 0, \ldots, 0)$$

is taken by σ to a convergent sequence, but the new limit is not $\sigma(1, 0, \ldots, 0) = (-1, 0, \ldots, 0)$.

Mon, Oct. 16

Last time, we discussed real projective space as a quotient $\mathbb{RP}^n \cong (\mathbb{R}^{n+1} - \{0\})/\mathbb{R}^\times$. We have a similar story for $\mathbb{CP}^n$.

Example 15.6. There is an action of $\mathbb{C}^\times$ on $\mathbb{C}^{n+1} \setminus \{0\}$, and the orbits are the punctured complex lines. We claim that the quotient is $\mathbb{CP}^n$.

We defined $\mathbb{CP}^n$ as a quotient of an S^1-action on S^{2n+1}. We also have a homeomorphism $\mathbb{C}^{n+1} \setminus \{0\} \cong S^{2n+1} \times \mathbb{R}_{>0}$ and an isomorphism $\mathbb{C}^\times \cong S^1 \times \mathbb{R}^\times_{>0}$. We can then describe $\mathbb{CP}^n$ as the two-step quotient

$$\big(\mathbb{C}^{n+1} \setminus \{0\}\big)/\mathbb{C}^\times \cong \Big((S^{2n+1} \times \mathbb{R}_{>0})/\mathbb{R}^\times_{>0}\Big)/S^1 \cong S^{2n+1}/S^1 = \mathbb{CP}^n.$$

We have been studying actions of topological groups on spaces, and the resulting quotient spaces X/G. But there is another way to think about this material. Suppose you have a set Y that you would like to topologize. One way to create a topology on Y is as follows. Pick a point $y_0 \in Y$. If there is a transitive action of some topological group G on Y, then the orbit-stabilizer theorem asserts that Y can be identified with G/H, where $H \leq G$ is the stabilizer subgroup consisting of all $h \in G$ such that $h \cdot y_0 = y_0$. But G/H is a topological space, so we define the topology on Y to be the one coming from the bijection $Y \cong G/H$.

Example 15.7. (Grassmannian) We saw that the projective spaces can be identified with the set of lines in $\mathbb{R}^n$ or $\mathbb{C}^n$, suitably topologized. We can similarly consider the set of k-dimensional linear subspaces in $\mathbb{R}^n$ (or $\mathbb{C}^n$). It is not clear how to topologize this set.

However, there is an action of $O(n)$ on the set of k-planes in $\mathbb{R}^n$. Really, this comes from an action of the larger group $Gl_n(\mathbb{R})$, but the $O(n)$-action turns out to be more convenient. Namely, if $A \in O(n)$ is an orthogonal matrix and $V \subseteq \mathbb{R}^n$ is a k-dimensional subspace, then $A(V) \subseteq \mathbb{R}^n$ is another k-dimensional subspace. Furthermore, this action is transitive. To see this, it suffices to show that given any subspace V, there is a matrix taking the standard subspace $E_k = \mathrm{Span}\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$ to V. Thus suppose $V = \mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a k-dimensional subspace with given orthonormal basis. This can be completed to an orthonormal basis of $\mathbb{R}^n$. Then if A is the orthogonal matrix with columns the $\mathbf{v}_i$, A takes the standard subspace E_k to V.

The stabilizer of E_k is the subgroup of orthogonal matrices that take E_k to E_k. Such matrices are block matrices, with an orthogonal $k \times k$ matrix in the upper left and an orthogonal $(n-k)\times(n-k)$ matrix in the lower right. In other words, the stabilizer subgroup is $O(k) \times O(n-k)$. It follows that the set of k-planes in $\mathbb{R}^n$ can be identified with the quotient

$$\mathrm{Gr}_{k,n}(\mathbb{R}) = O(n)/\Big(O(k) \times O(n-k)\Big).$$

Note that, from this identification, we can see that $\mathrm{Gr}_{k,n} \cong \mathrm{Gr}_{n-k,n}$. The map takes a k-plane in $\mathbb{R}^n$ to the orthogonal complement, which is an $n-k$-plane in $\mathbb{R}^n$. The corresponding map

$$O(n)/\Big(O(k) \times O(n-k)\Big) \longrightarrow O(n)/\Big(O(n-k) \times O(k)\Big)$$

is induced by a map $O(n) \longrightarrow O(n)$. This map on $O(n)$ is conjugation by a shuffle permutation that permutes k things past $n-k$ things.

There is an identical story for the complex Grasmannians, where $O(n)$ is replaced by $U(n)$.

Wed, Oct. 18

Part 4. **Properties**

16. Connectedness

What we have done so far corresponds roughly to Chapters 2 & 3 of Lee. Now we turn to Chapter 4.

The first idea is connectedness. Essentially, we want to say that a space cannot be decomposed into two disjoint pieces.

Definition 16.1. A **disconnection** (or separation) of a space X is a pair of disjoint, nonempty open subsets $U, V \subseteq X$ with $X = U \cup V$. We say that X is **connected** if it has no disconnection.

Example 16.2.
(1) If X is a discrete space (with at least two points), then any pair of disjoint nonempty subsets gives a disconnection of X.
(2) Let X be the subspace $(0,1) \cup (2,3)$ of $\mathbb{R}$. Then X is disconnected.
(3) More generally, if $X \cong A \coprod B$ for nonempty spaces A and B, then X is disconnected.
(4) Another example of a disconnected subspace of $\mathbb{R}$ is the subspace $\mathbb{Q}$. A disconnection of $\mathbb{Q}$ is given by $(-\infty, \pi) \cap \mathbb{Q}$ and $(\pi, \infty) \cap \mathbb{Q}$.
(5) Any set with the trivial topology is connected, since there is only one nonempty open set.
(6) Of the 29 topologies on $X = \{1, 2, 3\}$, 19 are connected, and the other 10 are disconnected. For example, the topology $\{\emptyset, \{1\}, X\}$ is connected, but $\{\emptyset, \{1\}, \{2, 3\}, X\}$ is not.
(7) If X is a space with the generic point (or included point) topology, in which the nonempty open sets are precisely the ones containing a special point x_0, then X is connected.
(8) If X is a space with the excluded point topology, in which the open proper subsets are the ones missing a special point x_0, then X is connected.
(9) The lower limit topology $\mathbb{R}_{\ell\ell}$ is disconnected, as the basis elements $[a, b)$ are both open and closed (clopen!), which means that their complements are open.

Proposition 16.3. *Let X be a space. The following are equivalent:*
(1) *X is disconnected*
(2) *$X \cong A \coprod B$ for nonempty spaces A and B*
(3) *There exists a nonempty, clopen, proper subset $U \subseteq X$*
(4) *There exists a continuous surjection $X \twoheadrightarrow \{0, 1\}$, where $\{0, 1\}$ has the discrete topology.*

Now let's look at an interesting example of a connected space.

Proposition 16.4. *The only (nonempty) connected subspaces of $\mathbb{R}$ are intervals (including singletons).*

Proof. Note that, by an interval, we mean simply a convex subset of $\mathbb{R}$. Any connected subset must be an interval since if A is connected and $a < b < c$ with $a, c \in A$, then either $b \in A$ or $(-\infty, b) \cap A$ and $(b, \infty) \cap A$ give a separation of A.

So it remains to show that intervals are connected. Singletons are connected, as there is only one nonempty subset. Thus let $I \subseteq \mathbb{R}$ be an interval with at least two points, and let $U \subseteq I$ be nonempty and clopen (in the subspace topology on I). We wish to show that $U = I$. Let $a \in U$. We will show that $U \cap [a, \infty) = I \cap [a, \infty)$. In other words, we wish to show that if $b > a$ and $b \in I$, then $b \in U$. A similar argument will show that $U \cap (-\infty, a] = I \cap (-\infty, a]$.

Consider the set
$$R_a = \{b \in I \mid [a, b] \subseteq U\}.$$
Note that $a \in R_a$, so that R_a is nonempty. If R_a is not bounded above, then $[a, \infty) \subseteq U \subseteq I$, and we have our conclusion.

Otherwise, the set R_a has a supremum $s = \sup R_a$ in $\mathbb{R}$. Note that if $s \notin I$, then since I is an interval, no real number larger y than s can be in I, since otherwise the entire interval $[a, y]$, which contains s, would be contained in I. Then

$$[a, s) \ \subseteq \ U \cap [a, \infty) \ \subseteq \ I \cap [a, \infty) = [a, s).$$

It follows that $U \cap [a, \infty) = I \cap [a, \infty) = [a, s)$.

The final case to consider is when $s \in I$. Since we can express s as a limit of a U-sequence and since U is closed in I, it follows that s must also lie in U. Since U is open, some ϵ-neighborhood of s (in I) lies in U. But no point in $(s, s + \epsilon/2)$ can lie in U (or I), since any such point would then also lie in R_a. Again, since I is an interval we have

$$U \cap [a, \infty) = [a, s] = I \cap [a, \infty).$$

■

Fri, Oct. 20

One of the most useful results about connected spaces is the following.

Proposition 16.5. *Let $f : X \longrightarrow Y$ be continuous. If X is connected, then so is $f(X) \subseteq Y$.*

Proof. Suppose that $U \subseteq f(X)$ is closed and open. Then $f^{-1}(U)$ must be closed and open, so it must be either $\emptyset$ or X. This forces $U = \emptyset$ or $U = f(X)$. ■

Since the exponential map $\exp : [0, 1] \longrightarrow S^1$ is a continuous surjection, it follows that S^1 is connected. More generally, we have

Proposition 16.6. *Let $q : X \longrightarrow Y$ be a quotient map with X connected. Then Y is connected.*

As another application, we have

Theorem 16.7 (Intermediate Value Theorem)**.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. Then f attains every intermediate value between $f(a)$ and $f(b)$.*

Proof. This follows from the fact that the image is connected and so must be an interval by Proposition 16.4. ■

Which of the other constructions we have seen preserve connectedness? All of them! (Well, except that subspaces of connected spaces need not be connected, as we have already seen.)

Proposition 16.8. *Let $A_i \subseteq X$ be connected for each i, and assume that $x_0 \in \bigcap_i A_i \neq \emptyset$. Then $\bigcup_i A_i$ is connected.*

Proof. Assume each A_i is connected, and let $U \subseteq \bigcup_i A_i$ be nonempty and clopen. Let $x \in U \subseteq \bigcup_i A_i$. Suppose $x \in A_{i_0}$. Then $U \cap A_{i_0}$ is nonempty and clopen in A_{i_0}, so $U \cap A_{i_0} = A_{i_0}$. In other words, $A_{i_0} \subseteq U$. Since $x_0 \in A_{i_0}$, it follows that $x_0 \in U$. But now for any other A_j, we have that $x_0 \in A_j \cap U$, so that $A_j \cap U$ is nonempty and clopen in A_j. It follows that $A_j \subseteq U$. ■

As an application, we get that products interact well with connectedness.

Proposition 16.9. *Assume $X_i \neq \emptyset$ for all $i \in \{1, \ldots, n\}$. Then $\displaystyle\prod_{i=1}^{n} X_i$ is connected if and only if each X_i is connected.*

Proof. ($\Rightarrow$) This follows from Prop 16.5, as $p_i : \prod_i X_i \longrightarrow X_i$ is surjective (this uses that all X_j are nonempty).

($\Leftarrow$) Suppose each X_i is connected. By induction, it suffices to show that $X_1 \times X_2$ is connected. Pick any $z \in X_2$. We then have the embedding $X_1 \hookrightarrow X_1 \times X_2$ given by $x \mapsto (x, z)$. Since X_1 is connected, so is its image C in the product. Now for each $x_1 \in X_1$, we have an embedding $\iota_{x_1} : X_2 \hookrightarrow X_1 \times X_2$ given by $y \mapsto (x_1, y)$. Let $D_{x_1} = \iota_{x_1}(X_2) \cup C$. Note that each D is connected, being the overlapping union of two connected subsets. But we can write $X_1 \times X_2$ as the overlapping union of all of the D_{x_1}, so by the previous result the product is connected. ∎

The following result is easy but useful.

Proposition 16.10. *Let $A \subseteq B \subseteq \overline{A}$ and suppose that A is connected. Then so is B.*

Proof. Exercise ∎

Theorem 16.11. *Assume $X_i \neq \emptyset$ for all $i \in I$, where is I is arbitrary. Then $\prod_i X_i$ is connected if and only if each X_i is connected.*

Proof. As in the finite product case, it is immediate that if the product is connected, then so is each factor.

We sketch the other implication. We have already showed that each finite product is connected. Now let $(z_i) \in \prod_i X_i$. For each $j \in I$, write $D_j = p_j^{-1}(z_j) \subseteq \prod_i X_i$.

For each finite collection $j_1, \ldots, j_k \in I$, let

$$F_{j_1,\ldots,j_k} = \bigcap_{j \neq j_1,\ldots,j_k} D_j \subseteq \prod_i X_i.$$

Then $F_{j_1,\ldots,j_k} \cong X_{j_1} \times \cdots \times X_{j_k}$, so it follows that $F_{j_1,\ldots,j_k}$ is connected. Now $(z_i) \in F_{j_1,\ldots,j_k}$ for every such tuple, so it follows that

$$F = \bigcup F_{j_1,\ldots,j_k}$$

is connected.

It remains to show that F is dense in $\prod_i X_i$ (in other words, the closure of F is the whole product). Let

$$U = p_{j_1}^{-1}(U_{j_1}) \cap \cdots \cap p_{j_k}^{-1}(U_{j_k})$$

be a nonempty basis element. Then U meets $F_{j_1,\ldots,j_k}$, so U meets F. Since U was arbitrary, it follows that F must be dense. ∎

Note that the above proof would not have worked with the box topology. We can show directly that $\mathbb{R}^{\mathbb{N}}$, equipped with the box topology, is not connected. Consider the subset $\mathcal{B} \subset \mathbb{R}^{\mathbb{N}}$ consisting of bounded sequences. If $(z_i) \in \mathcal{B}$, then $\prod_i (z_i - 1, z_i + 1)$ is a neighborhood of (z_i) in $\mathcal{B}$. On the other hand, if $(z_i) \notin \mathcal{B}$, the same formula gives a neighborhood consisting entirely of unbounded sequences. We conclude that $\mathcal{B}$ is a nontrivial clopen set in the box topology.

Mon, Oct. 23

16.1. **Path Connectedness.** Ok, so we have looked at examples and studied this notion of being connected, but if you asked your calculus students to describe what it should mean for a subset of $\mathbb{R}$ to be connected, they probably wouldn't come up with the "no nontrivial clopen subsets" idea. Instead, they would probably say something about being able to connect-the-dots. In other words, you should be able to draw a line from one point to another while staying in the subset. This leads to the following idea.

Definition 16.12. We say that $A \subseteq X$ is **path-connected** if for every pair a, b of points in A, there is a continuous function (a path) $\gamma : I \longrightarrow A$ with $\gamma(0) = a$ and $\gamma(1) = b$.

This is not unrelated to the earlier notion.

Proposition 16.13. *If $A \subseteq X$ is path-connected, then it is also connected.*

Proof. Pick a point $a_0 \in A$. For any other $b \in A$, we have a path γ_b in A from a_0 to b. Then the image $\gamma_b(I)$ is a connected subset of A containing both a_0 and b. It follows that

$$A = \bigcup_{b \in A} \gamma_b(I)$$

is connected, as it is the overlapping union of connected sets. ■

For subsets $A \subseteq \mathbb{R}$, we have

$$A \text{ is path-connected} \Rightarrow A \text{ is connected} \Leftrightarrow A \text{ is an interval} \Rightarrow A \text{ is path-connected.}$$

So the two notions coincide for subsets of $\mathbb{R}$. But the same is not true in $\mathbb{R}^2$! (The topologist's sine curve, HW 7).

Path-connectedness has much the same behavior as connectedness.

Proposition 16.14.

(1) *Images of path-connected spaces are path-connected*
(2) *Overlapping unions of path-connected spaces are path-connected*
(3) *Finite products of path-connected spaces are path-connected*

However, the topologist's sine curve shows that closures of path-connected subsets need not be path-connected.

Our proof of connectivity of $\prod_i X_i$ last time used this closure property for connected sets, so the earlier argument does not adapt easily to path-connectedness. But it turns out to be easier to prove.

Theorem 16.15. *Assume $X_i \neq \emptyset$ for all $i \in I$, where is I is arbitrary. Then $\prod_i X_i$ is path-connected if and only if each X_i is path-connected.*

Proof. The interesting direction is ($\Leftarrow$). Thus assume that each X_i is path-connected. Let (x_i) and (y_i) be points in the product $\prod_i X_i$. Then for each $i \in \mathcal{I}$ there is a path γ_i in X_i with $\gamma_i(0) = x_i$ and $\gamma_i(1) = y_i$. By the universal property of the product, we get a continuous path

$$\gamma = (\gamma_i) : [0,1] \longrightarrow \prod_i X_i$$

with $\gamma(0) = (x_i)$ and $\gamma(1) = (y_i)$. ■

16.2. **Components.** The overlapping union property for (path-)connectedness allows us to make the following definition.

Definition 16.16. Let $x \in X$. We define the **connected component** (or simply component) of x in X to be

$$C_x = \bigcup_{\substack{x \in C \\ \text{connected}}} C.$$

Similarly, the **path-component** of X is defined to be

$$PC_x = \bigcup_{\substack{x \in P \\ \text{connected}}} P.$$

The overlapping union property guarantees that C_x is connected and that PC_x is path-connected. Since path-connected sets are connected, it follows that for any x, we have $PC_x \subseteq C_x$. An immediate consequence of the above definition(s) is that any (path-)connected subset of X is contained in some (path-)component.

Example 16.17. Consider $\mathbb{Q}$, equipped with the subspace topology from $\mathbb{R}$. Then the only connected subsets are the singletons, so $C_x = \{x\}$. Such a space is said to be **totally disconnected**.

Note that for any space X, each component C_x is closed as $\overline{C_x}$ is a connected subset containing x, which implies $\overline{C_x} \subseteq C_x$. If X has finitely many components, then each component is the complement of the finite union of the remaining components, so each component is also open, and X decomposes as a disjoint union

$$X \cong C_1 \amalg C_2 \amalg \cdots \amalg C_n$$

of its components. But this does not happen in general, as the previous example shows.

The situation is worse for path-components: they need not be open or closed, as the topologist's sine curve shows.

16.3. **Locally (Path-)Connected.**

Definition 16.18. Let X be a space. We say that X is **locally connected** if any neighborhood U of any point x contains a connected neighborhood $x \in V \subset U$. Similarly X is **locally path-connected** if any neighborhood U of any point x contains a path-connected neighborhood $x \in V \subset U$.

The locally path-connected turns out to show up more often, so we focus on that.

Proposition 16.19. *Let X be a space. The following are equivalent.*

1. *X is locally path-connected*
2. *X has a basis consisting of path-connected open sets*
3. *for every open set $U \subseteq X$, the path-components of U are open in X*
4. *for every open set $U \subseteq X$, every component of U is path-connected and open in X.*

Proof. We leave the implications (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) as an exercise. We argue for (1) $\Leftrightarrow$ (4).

Assume X is locally path-connected, and let C be a component of an open subset $U \subseteq X$. Let $P \subseteq C$ be a nonempty path-component. Then P is open in X. But all of the other path-components of C are also open, so their union, which is the complement of P, must be open. It follows that P is closed. Since C is connected, we must have $P = C$.

On the other hand, suppose that (4) holds. Let U be a neighborhood of x. Then the component C_x of x in U is the desired neighborhood V. ■

In particular, this says that the components and path-components agree if X is locally path-connected.

Just as path-connected implies connected, locally path-connected implies locally-connected. But, unfortunately, there are no other implications between the four properties.

Example 16.20. The topologist's sine curve is connected, but not path-connected or locally connected or locally path-connected. Thus it is possible to be connected but not locally so.

Example 16.21. For any space X, the **cone** on X is defined to be $CX = X \times [0,1]/X \times \{1\}$. The cone on any space is always path-connected. In particular, the cone on the topologist's sine curve is connected and-path connected but not locally connected or locally path-connected.

Example 16.22. A disjoint union of two topologist's sine curves gives an example that is not connected in any of the four ways.

Example 16.23. Note that if X is locally path-connected, then connectedness is equivalent to path-connectedness. A connected example would be $\mathbb{R}$ or a one-point space. A disconnected example would be $(0,1) \cup (2,3)$ or a two point (discrete) space.

Finally, we have spaces that are locally connected but not locally path-connected.

Example 16.24. The cocountable topology on $\mathbb{R}$ is connected and locally connected but not path-connected or locally path-connected.

Example 16.25. The cone on the cocountable topology will give a connected, path-connected, locally connected space that is not locally path-connected.

Example 16.26. Two copies of $\mathbb{R}_{\text{cocountable}}$ give a space that is locally connected but not connected in the other three ways.

Wed, Oct. 25

17. Compactness

The next topic is one of the major ones in the course: compactness. As we will see, this is the analogue of a "closed and bounded subset" in a general space. The definition relies on the idea of coverings.

Definition 17.1. An **open cover** of X is a collection $\mathcal{U}$ of open subsets that cover X. In other words, $\bigcup_{U \in \mathcal{U}} U = X$. Given two covers $\mathcal{U}$ and $\mathcal{V}$ of X, we say that $\mathcal{V}$ is a **subcover** if $\mathcal{V} \subseteq \mathcal{U}$.

Definition 17.2. A space X is said to be **compact** if every open cover has a *finite* subcover (i.e. a cover involving finitely many open sets).

Example 17.3. Clearly any finite topological space is compact, no matter the topology.

Example 17.4. An infinite set with the discrete topology is *not* compact, as the collection of singletons gives an open cover with no finite subcover.

Example 17.5. $\mathbb{R}$ is not compact, as the open cover $\mathcal{U} = \{(-n,n) \mid n \in \mathbb{N}\}$ has no finite subcover.

Example 17.6. Similarly $[0,\infty)$ is not compact, as the open cover $\mathcal{U} = \{[0,n)\}$ has no finite subcover. Recall that $[0,\infty) \cong [a,b)$.

Theorem 17.7. *Let $a < b$. Then $[a,b]$ is a compact subset of $\mathbb{R}$.*

Proof. Let $\mathcal{U}$ be an open cover. Then some element of the cover must contain a. Pick such an element and call it U_0.

Consider the set

$$\mathcal{E} = \{c \in [a,b] \mid [a,c] \text{ is finitely covered by } \mathcal{U}\}.$$

Certainly $a \in \mathcal{E}$ and $\mathcal{E}$ is bounded above by b. By the Least Upper Bound Axiom, $s = \sup \mathcal{E}$ exists. Note that $a \leq s \leq b$, so we must have $s \in U_s$ for some $U_s \in \mathcal{U}$. In general, U_s may not be connected, so let $s \in V \subseteq U_s$ be an open interval. But then for any $c < s$ with $c \in V \subseteq U_s$, we have $c \in \mathcal{E}$. This means that

$$[a,c] \subseteq U_1 \cup \cdots \cup U_k$$

for $U_1, \ldots, U_k \in \mathcal{U}$. But then $[a,s] \subseteq U_1 \cup \cdots \cup U_k \cup U_s$. This shows that $s \in \mathcal{E}$. On the other hand, the same argument shows that for any $s < d < b$ with $d \in U_s$, we would similarly have $d \in \mathcal{E}$. Since $s = \sup \mathcal{E}$, there cannot exist such a d. This implies that $s = b$. ■

Like connectedness, compactness is preserved by continuous functions.

Proposition 17.8. *Let $f : X \longrightarrow Y$ be continuous, and assume that X is compact. Then $f(X)$ is compact.*

Proof. Let $\mathcal{V}$ be an open cover of $f(X)$. Then $\mathcal{U} = \{f^{-1}(V) \mid V \in \mathcal{V}\}$ is an open cover of X. Let $\{U_1, \ldots, U_k\}$ be a finite subcover. It follows that the corresponding $\{V_1, \ldots, V_k\}$ is a finite subcover of $\mathcal{V}$. ■

Fri, Oct 27

Example 17.9. Recall that we have the quotient map $\exp : [0,1] \longrightarrow S^1$. It follows that S^1 is compact.

Theorem 17.10 (Extreme Value Theorem)**.** *Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous. Then f attains a maximum and a minimum.*

Proof. Since f is continuous and $[a,b]$ is both connected and compact, the same must be true of its image. But the compact, connected subsets are precisely the closed intervals. ■

The following result is also quite useful.

Proposition 17.11. *Let X be Hausdorff and let $A \subseteq X$ be a compact subset. Then A is closed in X.*

Proof. Pick any point $x \in X \setminus A$ (if we can't, then $A = X$ and we are done). For each $a \in A$, we have disjoint neighborhoods $a \in U_a$ and $x \in V_a$. Since the U_a cover A, we only need finitely many, say $U_{a_1}, \ldots, U_{a_k}$ to cover A. But then the intersection

$$V = V_{a_1} \cap \cdots \cap V_{a_k}$$

of the corresponding V_a's is disjoint from the union of the U_a's and therefore also from A. Since V is a finite intersection of open sets, it is open and thus gives a neighborhood of x in $X \setminus A$. It follows that A is closed. ■

Exercise 17.12. *If $A \subseteq X$ is closed and X is compact, then A is compact.*

Combining these results gives the following long-awaited consequence.

Corollary 17.13. *Let $f : X \longrightarrow Y$ be continuous, where X is compact and Y is Hausdorff, then f is a closed map.*

In particular, if f is already known to be a continuous bijection, then it is automatically a homeomorphism. For example, this shows that the map $I/\partial I \longrightarrow S^1$ is a homeomorphism. Similarly, from Example 14.10 we have $D^n/\partial D^n \cong S^n$.

17.1. **Products.** We will next show that finite products of compact spaces are compact, but we first need a lemma.

Lemma 17.14 (Tube Lemma)**.** *Let X be compact and Y be any space. If $W \subseteq X \times Y$ is open and contains $X \times \{y\}$, then there is a neighborhood V of y with $X \times V \subseteq W$.*

Proof. For each $x \in X$, we can find a basic neighborhood $U_x \times V_x$ of (x, y) in W. The U_x's give an open cover of X, so we only need finitely many of them, say $U_{x_1}, \ldots, U_{x_n}$. Then we may take $V = V_{x_1} \cap \cdots \cap V_{x_n}$. ■

Mon, Oct. 30

Proposition 17.15. *Let X and Y be nonempty. Then $X \times Y$ is compact if and only if X and Y are compact.*

Proof. As for connectedness, the continuous projections make X and Y compact if $X \times Y$ is compact.

Now suppose that X and Y are compact and let $\mathcal{U}$ be an open cover. For each $y \in Y$, the cover $\mathcal{U}$ of $X \times Y$ certainly covers the slice $X \times \{y\}$. This slice is homeomorphic to X and therefore finitely-covered by some $\mathcal{V} \subset \mathcal{U}$. By the Tube Lemma, there is a neighborhood V_y of y such that the tube $X \times V_y$ is covered by the same $\mathcal{V}$. Now the V_y's cover Y, so we only need finitely many of these to cover X. Since each tube is finitely covered by $\mathcal{U}$ and we can cover $X \times Y$ by finitely many tubes, it follows that $\mathcal{U}$ has a finite subcover. ■

17.2. **Compactness in $\mathbb{R}^n$.**

Theorem 17.16 (Heine-Borel)**.** *A subset $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded (contained in a single metric ball).*

Proof. Suppose A is compact. Then A must be closed in $\mathbb{R}^n$ since $\mathbb{R}^n$ is Hausdorff. To see that A is bounded, pick any point $a \in A$ (if A is empty, we are certainly done). Then the collection of balls $B_n(a) \cap A$ gives an open cover of A, since any other point in A is a finite distance away from a. Since A is compact, there must be a finite subcover $\{B_{n_1}(a), \dots, B_{n_k}(a)\}$. Let $N = \max\{n_1, \dots, n_k\}$. Then $A \subseteq B_N(a)$.

On the other hand, suppose that A is closed and bounded in $\mathbb{R}^n$. Since A is bounded, it is contained in $[-k, k]^n$ for some $k > 0$. But this product of intervals is compact since each interval is compact. Now A is a closed subset of a compact space, so it is compact. ■

In fact, the forward implication of the above proof works to show that

Proposition 17.17. *Let $A \subseteq X$, where X is metric and A is compact. Then A is closed and bounded in X.*

But the reverse implication is not true in general, as the next example shows.

Example 17.18. Consider $[0, \pi] \cap \mathbb{Q} \subseteq \mathbb{Q}$. This is certainly closed and bounded, but we will see it is not compact. Consider the open cover $\mathcal{U} = \left\{[0, \pi - \frac{1}{n}) \cap \mathbb{Q}\right\}_{n \in \mathbb{N}}$. This has no finite subcover.

Again, we have shown that compactness interacts well with finite products, and we would like a similar result in the arbitrary product case. This is a major theorem, known as the Tychonoff theorem. First, we show the theorem does not hold with the box topology.

Example 17.19. Consider $X = \{0, 1\}^{\mathbb{N}}$. In the box topology, this is discrete. Since this is infinite, it cannot be compact.

Example 17.20. We have studied the orthogonal subgroups $O(n) \subseteq Gl_n(\mathbb{R})$. The bigger group $Gl_n(\mathbb{R})$ is not compact, as it is neither closed nor bounded in $\mathbb{R}^{n^2}$. However, the orthogonality relations defining orthogonal matrices make this a closed subset of $\mathbb{R}^{n^2}$, and the fact that each column has norm 1 means that an orthogonal $n \times n$ matrix, when considered as a point in $\mathbb{R}^{n^2}$, has norm $\sqrt{n}$. In particular, $O(n)$ is a bounded subset of $\mathbb{R}^{n^2}$.

17.3. **Tychonoff's Theorem.**

Theorem 17.21 (Tychonoff)**.** *Let $X_i \neq \emptyset$ for all $i \in \mathcal{I}$. Then $\prod_i X_i$ is compact if and only if each X_i is compact.*

Our proof, even for the difficult direction, will use the axiom of choice. In fact, Tychonoff's theorem is equivalent to the axiom of choice.

Theorem 17.22. *Tychonoff $\Rightarrow$ axiom of choice.*

Proof. This argument is quite a bit simplier than the other implication. Let $X_i \neq \emptyset$ for all $i \in \mathcal{I}$. We want to show that $X = \prod_i X_i \neq \emptyset$.

For each i, define $Y_i = X_i \cup \{\infty_i\}$, where $\infty_i \notin X_i$. We topologize Y_i such that the only nontrivial open sets are X_i and $\{\infty_i\}$. Now for each i, let $U_i = p_i^{-1}(\infty_i)$. The collection $\mathcal{U} = \{U_i\}$ gives a collection of open subsets of $Y = \prod_i Y_i$, and this collection covers Y if and only if $X = \emptyset$. Each Y_i is compact since it has only four open sets. Thus Y must be compact by the Tychonoff theorem. But no finite subcollection of $\mathcal{U}$ can cover Y. For example, $U_i \cup U_j$ does not cover Y since if $a \in X_i$ and $b \in X_j$, then we can define $(y_i) \in Y \setminus (U_i \cup U_j)$ by

$$y_k = \begin{cases} a & k = i \\ b & k = j \\ \infty_k & k \neq i, j \end{cases}$$

The same kind of argument will work for any finite collection of U_i's. Since $\mathcal{U}$ has no finite subcover and Y is compact, $\mathcal{U}$ cannot cover Y, so that X must be nonempty. ■

Wed, Nov. 1

The Tychonoff Theorem is *equivalent* to the axiom of choice. We will thus use a form of the axiom of choice in order to prove it.

Zorn's Lemma. Let P be a partially-ordered set. If every linearly-ordered subset of P has an upper bound in P, then P contains at least one maximal element.

Theorem 17.23 (Tychonoff)**.** *Let $X_i \neq \emptyset$ for all $i \in \mathcal{I}$. Then $\prod_i X_i$ is compact if and only if each X_i is compact.*

Proof. As we have seen a number of times, the implication ($\Rightarrow$) is trivial.

We now show the contrapositive of ($\Leftarrow$). Thus assume that $X = \prod_i X_i$ is not compact. We wish to conclude that one of the X_i must be noncompact. By hypothesis, there exists an open cover $\mathcal{U}$ of X with no finite subcover.

We first deal with the following case.

Special case: *$\mathcal{U}$ is a cover by prebasis elements.*
For each $i \in \mathcal{I}$, let $\mathcal{U}_i$ be the collection

$$\mathcal{U}_i = \{V \subseteq X_i \text{ open } \mid p_i^{-1}(V) \in \mathcal{U}\}.$$

For some i, the collection $\mathcal{U}_i$ must cover X_i, since otherwise we could pick $x_i \in X_i$ for each i with x_i not in the union of $\mathcal{U}_i$. Then the element $(x_i) \in \prod_i X_i$ would not be in $\mathcal{U}$ since it cannot be in any $p_i^{-1}(V)$. But now the cover $\mathcal{U}_i$ cannot have a finite subcover, since that would provide a corresponding subcover of $\mathcal{U}$. It follows that X_i is not compact.

It remains to show that we can always reduce to the cover-by-prebasis case.

Consider the collection $\mathcal{N}$ of open covers of X having no finite subcovers. By assumption, this set is nonempty, and we can partially order $\mathcal{N}$ by inclusion of covers. Furthermore, if $\{\mathcal{U}_\alpha\}$ is a linearly order subset of $\mathcal{N}$, then $\mathcal{U} = \bigcup_\alpha \mathcal{U}_\alpha$ is an open cover, and it cannot have a finite subcover since a finite subcover of $\mathcal{U}$ would be a finite subcover of one of the $\mathcal{U}_\alpha$. Thus $\mathcal{U}$ is an upper bound in $\mathcal{N}$ for $\{\mathcal{U}_\alpha\}$. By Zorn's Lemma, $\mathcal{N}$ has a maximal element $\mathcal{V}$.

Now let $\mathcal{S} \subseteq \mathcal{V}$ be the subcollection consisting of the prebasis elements in $\mathcal{V}$. We claim that $\mathcal{S}$ covers X. *Suppose not.* Thus let $x \in X$ such that x is not covered by $\mathcal{S}$. Then x must be in V for some $V \in \mathcal{V}$. By the definition of the product topology, x must have a basic open neighborhood in $B \subset V$. But any basic open set is a finite intersection of prebasic open sets, so $B = S_1 \cap \dots S_k$. If x is not covered by $\mathcal{S}$, then none of the S_i are in $\mathcal{S}$. Thus $\mathcal{V} \cup \{S_i\}$ is not in $\mathcal{N}$ by maximality of $\mathcal{V}$. In other words, $\mathcal{V} \cup \{S_i\}$ has a finite subcover $\{V_{i,1}, \dots, V_{i,n_i}, S_i\}$. Let us write

$$\hat{V}_i = V_{i,1} \cup \cdots \cup V_{i,n_i}.$$

Now

$$X = \bigcap_i \left(S_i \cup \hat{V}_i\right) \subseteq \left(\bigcap_i S_i\right) \cup \left(\bigcup_i \hat{V}_i\right) \subseteq V \cup \left(\bigcup_i \hat{V}_i\right)$$

This shows that $\mathcal{V}$ has a finite subcover, which contradicts that $\mathcal{V} \in \mathcal{N}$. We thus conclude that $\mathcal{S}$ covers X using only prebasis elements.

But now by the argument at the beginning of the proof, $\mathcal{S}$, and therefore $\mathcal{V}$ as well, has a finite subcover. This is a contradiction. ■

Fri, Nov. 3

Remark 17.24. There are other versions of compactness. For instance **sequential compactness** is the condition that every sequence has a convergent subsequence. In a metric space, this turns out to be equivalent to compactness, but not for general topological spaces.

17.4. Local Compactness.

Definition 17.25. We say that a space is **locally compact** if every $x \in X$ has a compact neighborhood (recall that we do not require neighborhoods to be open).

This looks different from our other "local" notions. To get a statement in the form we expect, we introduce more terminology $A \subseteq X$ is **precompact** if $\overline{A}$ is compact.

Proposition 17.26. *Let X be Hausdorff. TFAE*

(1) *X is locally compact*
(2) *every $x \in X$ has a precompact neighborhood*
(3) *X has a basis of precompact open sets*

Proof. It is clear that (3) ⇒ (2) ⇒ (1) without the the Hausdorff assumption, so we show that (1) ⇒ (3). Suppose X is locally compact and Hausdorff. Let V be open in X and let $x \in V$. We want a precompact open neighborhood of x in V. Since X is locally compact, we have a compact neighborhood K of x, and since X is Hausdorff, K must be closed. Since V and K are both neighborhoods of x, so is $V \cap K$. Thus let $x \in U \subseteq V \cap K$. Then $\overline{U} \subseteq K$ since K is closed, and $\overline{U}$ is compact since it is a closed subset of a compact set. ■

In contrast to the local connectivity properties, it is clear that any compact space is locally compact. But this is certainly a generalization of compactness, since *any interval* in $\mathbb{R}$ is locally compact.

Example 17.27. A standard example of a space that is not locally compact is $\mathbb{Q} \subseteq \mathbb{R}$. We show that 0 does not have any compact neighborhoods . Let V be any neighborhood of 0. Then it must contain $(-\pi/n, \pi/n)$ for some n. Now

$$\mathcal{U} = \left\{ \left(-\pi/n, \left(\frac{k}{k+1}\right)\pi/n \right) \right\} \cup \left\{ V \cap (\pi/n, \infty), V \cap (-\infty, -\pi/n) \right\}$$

is an open cover of V with no finite subcover.

Remark 17.28. Why did we define local compactness in a different way from local (path)-connectedness? We could have defined locally connected to mean that every point has a connected neighborhood , which follows from the actual definition. But then we would not have that locally connected is equivalent to having a basis of connected open sets. On the other hand, we could try the $x \in K \subseteq U$ version of locally compact, but of course we don't want to allow $K = \{x\}$, so the next thing to require is $x \in V \subseteq U$, where V is precompact. As we showed in Prop 17.26, this is equivalent to our definition of locally compact in the presence of the Hausdorff condition. Without the Hausdorff condition, compactness does not behave quite how we expect.

18. Compactification

Locally compact Hausdorff spaces are a very nice class of spaces (almost as good as compact Hausdorff). In fact, any such space is close to a compact Hausdorff space.

Definition 18.1. A **compactification** of a noncompact space X is an embedding $i : X \hookrightarrow Y$, where Y is compact and $i(X)$ is dense.

We will typically work with Hausdorff spaces X, in which case we ask the compactification Y to also be Hausdorff.

Example 18.2. The open interval $(0, 1)$ is not compact, but $(0, 1) \hookrightarrow [0, 1]$ is a compactification. Note that the exponential map $\exp : (0, 1) \longrightarrow S^1$ also gives a (different) compactification. The topologist's sine curve (HW 7.5) also gives a (much larger) compactification.

Mon, Nov. 6

There is often a smallest compactification, given by the following construction.

Definition 18.3. Let X be a space and define $\widehat{X} = X \cup \{\infty\}$, where $U \subseteq \widehat{X}$ is open if either

- $U \subseteq X$ and U is open in X or
- $\infty \in U$ and $\widehat{X} \setminus U \subseteq X$ is compact.

Proposition 18.4. *Suppose that X is Hausdorff and noncompact. Then $\widehat{X}$ is a compactification. If X is locally compact, then $\widehat{X}$ is Hausdorff.*

Proof. We first show that $\widehat{X}$ is a space! It is clear that both $\emptyset$ and $\widehat{X}$ are open.

Suppose that U_1 and U_2 are open. We wish to show that $U_1 \cap U_2$ is open.

- If neither open set contains ∞, this follows since X is a space.
- If $\infty \in U_1$ but $\infty \notin U_2$, then $K_1 = X \setminus U_1$ is compact. Since X is Hausdorff, K_1 is closed in X. Thus $X \setminus K_1 = U_1 \setminus \{\infty\}$ is open in X, and it follows that $U_1 \cap U_2 = (U_1 \setminus \{\infty\}) \cap U_2$ is open.
- If $\infty \in U_1 \cap U_2$, then $K_1 = X \setminus U_1$ and $K_2 = X \setminus U_2$ are compact. It follows that $K_1 \cup K_2$ is compact, so that $U_1 \cap U_2 = X \setminus (K_1 \cup K_2)$ is open.
- Suppose we have a collection U_i of open sets. If none contain ∞, then neither does $\bigcup_i U_i$, and the union is open in X. If $\infty \in U_j$ for some j, then $\infty \in \bigcup_i U_i$ and
$$\widehat{X} \setminus \bigcup_i U_i = \bigcap_i (\widehat{X} \setminus U_i) = \bigcap_i (X \setminus U_i)$$
is a closed subset of the compact set $X \setminus U_j$, so it must be compact.

Next, we show that $\iota : X \longrightarrow \widehat{X}$ is an embedding. Continuity of ι again uses that compact subsets of X are closed. That ι is open follows immediately from the definition of $\widehat{X}$.

To see that $\iota(X)$ is dense in $\widehat{X}$, it suffices to see that $\{\infty\}$ is not open. But this follows from the definition of $\widehat{X}$, since X is not compact.

Finally, we show that $\widehat{X}$ is compact. Let $\mathcal{U}$ be an open cover. Then some $U \in \mathcal{U}$ must contain ∞. The remaining elements of $\mathcal{U}$ must cover $X \setminus U$, which is compact. It follows that we can cover $X \setminus U$ using only finitely many elements, so $\mathcal{U}$ has a finite subcover.

Now suppose that X is locally compact. Let x_1 and x_2 in $\widehat{X}$. If neither is ∞, then we have disjoint neighborhoods in X, and these are still disjoint neighborhoods in $\widehat{X}$. If $x_2 = \infty$, let $x_1 \in U \subseteq K$, where U is open and K is compact. Then U and $V = \widehat{X} \setminus K$ are the desired disjoint neighborhoods. ■

Example 18.5. We saw that S^1 is a one-point compactification of $(0, 1) \cong \mathbb{R}$. You will show on your homework that similarly S^n is a one-point compactification of $\mathbb{R}^n$.

Example 18.6. As we have seen, $\mathbb{Q}$ is not locally compact, so we do not expect $\widehat{\mathbb{Q}}$ to be Hausdorff. Indeed, any open subset containing ∞ is dense in $\widehat{\mathbb{Q}}$. Because of the topology on $\widehat{\mathbb{Q}}$, this is equivalent to showing that for any open, nonempty subset $U \subseteq \mathbb{Q}$, U is not contained in any compact subset. Since $\mathbb{Q}$ is Hausdorff, if U were contained in a compact subset, then $\overline{U}$ would also be compact. But as we have seen, for any interval $(a, b) \cap \mathbb{Q}$, the closure in $\mathbb{Q}$, which is $[a, b] \cap \mathbb{Q}$, is not compact.

Next, we show that the situation we observed for compactifications of $(0, 1)$ holds quite generally.

$$\begin{tikzcd} Y \arrow[rr, dashed, "q"] & & \widehat{X} \\ & X \arrow[ul, "f"] \arrow[ur, "\iota"'] & \end{tikzcd}$$

Proposition 18.7. *Let X be locally compact Hausdorff and let $f : X \longrightarrow Y$ be a (Hausdorff) compactification. Then there is a (unique) quotient map $q : Y \longrightarrow \widehat{X}$ such that $q \circ f = \iota$.*

We will need:

Lemma 18.8. *Let X be locally compact Hausdorff and $f : X \longrightarrow Y$ a compactification. Then f is open.*

Proof of Prop. 18.7. We define

$$q(y) = \begin{cases} \iota(x) & \text{if } y = f(x) \\ \infty & \text{if } y \notin f(X). \end{cases}$$

To see that q is continuous, let $U \subseteq \widehat{X}$ be open. If $\infty \notin U$, then $q^{-1}(U) = f(\iota^{-1}(U))$ is open by the lemma. If $\infty \in U$, then $K = \widehat{X} \setminus U$ is compact and thus closed. We have $q^{-1}(K) = f(\iota^{-1}(K))$ is compact and closed in Y, so it follows that $q^{-1}(U) = Y \setminus q^{-1}(K)$ is open.

Note that q is automatically a quotient map since it is a closed continuous surjection (it is closed because Y is compact and $\widehat{X}$ is Hausdorff). Note also that q is unique because $\widehat{X}$ is Hausdorff and q is already specified on the dense subset $f(X) \subseteq Y$. ■

Wed, Nov. 8

Last time, we said that we had a *unique* quotient map $q : Y \longrightarrow \widehat{X}$ for any Hausdorff compactification Y. Why is it unique? The definition of q on the dense subset $f(X) \subset Y$ was forced, and $\widehat{X}$ is Hausdorff. Then uniqueness is given by

Proposition 18.9. *Let Z be Hausdorff, and let $f, g : X \rightrightarrows Z$ be continuous functions. If f and g agree on a dense subset, then they agree on all of X.*

Proof of Lemma. Since f is an emebedding, we can pretend that $X \subseteq Y$ and that f is simply the inclusion. We wish to show that X is open in Y. Thus let $x \in X$. Let U be a precompact neighborhood of x. Thus $K = \mathrm{cl}_X(U)$ is compact[3] and so must be closed in Y (and X) since Y is Hausdorff. By the definition of the subspace topology, we must have $U = V \cap X$ for some open $V \subseteq Y$. Then V is a neighborhood of x in Y, and

$$V = V \cap Y = V \cap \mathrm{cl}_Y(X) \subseteq \mathrm{cl}_Y(V \cap X) = K \subseteq X.$$

The middle inclusion can be checked using the neighborhood criterion, using that V is open in Y. ■

Corollary 18.10. *Any two one-point compactifications are homeomorphic.*

The following is a useful characterization of locally compact Hausdorff spaces.

Proposition 18.11. *A space X is Hausdorff and locally compact if and only if it is homeomorphic to an open subset of a compact Hausdorff space Y.*

Proof. ($\Rightarrow$). We saw that X is open in the compact Hausdorff space $Y = \widehat{X}$.

($\Leftarrow$) As a subspace of a Hausdorff space, X must also be Hausdorff. It remains to show that every point has a compact neighborhood (in X). Write $Y_\infty = Y \setminus X$. This is closed in Y and therefore compact. By Problem 3 from HW7, we can find disjoint open sets $x \in U$ and $Y_\infty \subseteq V$ in Y. Then $K = Y \setminus V$ is the desired compact neighborhood of x in X. ■

[3]We will need to distinguish between closures in X and closures in Y, so we use the notation $\mathrm{cl}_X(A)$ for closure rather than our usual $\overline{A}$.

Corollary 18.12. *If X and Y are locally compact Hausdorff, then so is $X \times Y$.*

Corollary 18.13. *Any open or closed subset of a locally compact Hausdorff space is locally compact Hausdorff.*

18.1. **Separation Axioms.** We finally turn to the so-called "separation axioms".

Definition 18.14. A space X is said to be

- T_0 if given two distinct points x and y, there is a neighborhood of one not containing the other
- T_1 if given two distinct points x and y, there is a neighborhood of x not containing y and vice versa (points are closed)
- T_2 (**Hausdorff**) if any two distinct points x and y have disjoint neighborhoods
- T_3 (**regular**) if *points are closed and* given a closed subset A and $x \notin A$, there are disjoint open sets U and V with $A \subseteq U$ and $x \in V$
- T_4 (**normal**) if *points are closed and* given closed disjoint subsets A and B, there are disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.

Note that $T_4 \implies T_3 \implies T_2 \implies T_1 \implies T_0$. But beware that in some literature, the "points are closed" clause is not included in the definition of regular or normal. Without that, we would not be able to deduce T_2 from T_3 or T_4.

We have talked a lot about Hausdorff spaces. The other important separation property is T_4. We will not really discuss the intermediate notion of regular (or the other variants completely regular, completely normal, etc.)

Proposition 18.15. *Any compact Hausdorff space is normal.*

Proof. This was homework problem 8.5. ∎

More generally,

Theorem 18.16. *Suppose X is locally compact, Hausdorff, and second-countable. Then X is normal.*

Another important class of normal spaces is the collection of metric spaces.

Proposition 18.17. *If X is metric, then it is normal.*

Unfortunately, the T_4 condition alone is not preserved by the constructions we have studied.

Example 18.18. (Images) $\mathbb{R}$ is normal. But recall the quotient map $q : \mathbb{R} \longrightarrow \{-1, 0, 1\}$ which sends any number to its sign. This quotient is not Hausdorff and therefore not (regular or) normal.

Example 18.19. (Subspaces) If J is uncountable, then the product $(0,1)^J$ is *not* normal (Munkres, example 32.2). This is a subspace of $[0,1]^J$, which is compact Hausdorff by the Tychonoff theorem and therefore normal. So a subspace of a normal space need not be normal. We also saw in this example that (uncountable) products of normal spaces need not be normal.

Ok, so we've seen a few examples. So what, why should we care about normal spaces? Look back at the definition for T_2, T_3, T_4. In each case, we need to find certain open sets U and V. How would one do this in general? In a metric space, we would build these up by taking unions of balls. In an arbitrary space, we might use a basis. But another way of getting open sets is by pulling back open sets under a continuous map. That is, suppose we have a map $f : X \longrightarrow [0,1]$ such that $f \equiv 0$ on A and $f \equiv 1$ on B. Then $A \subseteq U := f^{-1}([0, \frac{1}{2}))$ and $B \subseteq V := f^{-1}((\frac{1}{2}, 1])$, and $U \cap V = \emptyset$.

One of the main consequences of normality is

Theorem 18.20 (Urysohn's Lemma)**.** *Let X be normal and let A and B be disjoint closed subsets. Then there exists a continuous function $f : X \longrightarrow [0, 1]$ such that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$.*

Note that Urysohn's Lemma becomes an if and only if statement if we either drop the T_1-condition from normal or if we explicitly include singletons as possible replacements for A and B.

A typical application of Urysohn's lemma is to create **bump functions**, which are equal to 1 on a closed set A and vanish outside some open $U \supset A$.

Theorem 18.21. *Suppose X is locally compact, Hausdorff, and second-countable. Then X is metrizable.*

See [Munkres, Theorem 34.1]. The point is that you can use Urysohn functions to give an embedding of X into $\mathbb{R}^{\mathbb{N}}$.

Part 5. **Nice spaces - the ones we really, really care about**

Fri, Nov. 10

19. Manifolds

We finally arrive at one of the most important definitions of the course.

Definition 19.1. A (topological) n**-manifold** M is a Hausdorff, second-countable space such that each point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

Example 19.2. (1) $\mathbb{R}^n$ and any open subset is obviously an n-manifold

(2) S^1 is a 1-manifold. More generally, S^n is an n-manifold. Indeed, we have shown that if you remove a point from S^n, the resulting space is homeomorphic to $\mathbb{R}^n$.

(3) T^n, the n-torus, is an n-manifold. In general, if M is an m-manifold and N is an n-manifold, then $M \times N$ is an $(m + n)$-manifold.

(4) $\mathbb{RP}^n$ is an n-manifold. There is a standard covering of $\mathbb{RP}^n$ by open sets as follows. Recall that $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^\times$. For each $1 \le i \le n+1$, let $V_i \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ be the complement of the hyperplane $x_i = 0$. This is an open, saturated set, and so its image $U_i = V_i/\mathbb{R}^\times \subseteq \mathbb{RP}^n$ is open. The V_i's cover $\mathbb{R}^{n+1} \setminus \{0\}$, so the U_i's cover $\mathbb{RP}^n$. We leave the rest of the details as an exercise.

(5) $\mathbb{CP}^n$ is a $2n$-manifold. This is similar to the description given above.

(6) $O(n)$ is a $\frac{n(n-1)}{2}$-manifold. Since it is also a topological group, this makes it a *Lie group*. The standard way to see that this is a manifold is to realize the orthogonal group as the preimage of the identity matrix under the transformation $M_n(R) \longrightarrow M_n(R)$ that sends A to $A^T A$. This map lands in the subspace $S_n(R)$ of symmetric $n \times n$ matrices. This space can be identified with $\mathbb{R}^{n(n+1)/2}$.

Now the $n \times n$ identity matrix is an element of S_n, and an important result in differential topology (Sard's theorem) that says that if a certain derivative map is surjective, then the preimage of the submanifold $\{I_n\}$ will be a submanifold of $M_n(\mathbb{R})$ of the same "codimension". In this case, the relevant derivative is the matrix of partial derivatives of $A \mapsto A^T A$, writen in a suitable basis. It follows that
$$\dim O(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$
The dimension statement can also be seen directly as follows. If A is an orthogonal matrix, its first column is just a point of S^{n-1}. Then its second column is a point on the sphere

orthogonal to the first column, so it lives in an "equator", meaning a sphere of dimension one less. Continuing in this way, we see that the "degree of freedom" for specifying a point of $O(n)$ is $(n-1)+(n-2)+\cdots+1=\frac{n(n-1)}{2}$.

(7) $\mathrm{Gr}_{k,n}(\mathbb{R})$ is a $k(n-k)$-manifold. One way to see this is to use the homeomorphism

$$\mathrm{Gr}_{k,n}(\mathbb{R}) \cong O(n)/\big(O(k)\times O(n-k)\big)$$

from Example 15.7. We get

$$\begin{aligned}\dim \mathrm{Gr}_{n,k}(\mathbb{R}) &= \dim O(n) - \big(\dim O(k) + \dim O(n-k)\big)\\ &= \sum_{j=1}^{n-1} j - \left(\sum_{j=1}^{k-1} j + \sum_{\ell=1}^{n-k-1} \ell\right) = \sum_{j=k}^{n-1} j - \sum_{\ell=1}^{n-k-1} \ell\\ &= \sum_{\ell=0}^{n-k-1} k+\ell - \sum_{\ell=0}^{n-k-1} \ell = \sum_{\ell=0}^{n-k-1} k = k(n-k)\end{aligned}$$

Here are some nonexamples of manifolds.

Example 19.3. (1) The union of the coordinate axes in $\mathbb{R}^2$. Every point has a neighborhood like $\mathbb{R}^1$ except for the origin.

(2) A discrete uncountable set is not second countable.

(3) A 0-manifold is discrete, so $\mathbb{Q}$ is not a 0-manifold.

(4) Glue together two copies of $\mathbb{R}$ by identifying any nonzero x in one copy with the point x in the other. This is second-countable and looks locally like $\mathbb{R}^1$, but it is not Hausdorff.

19.1. Properties of Manifolds.

Proposition 19.4. *Any manifold is locally path-connected.*

This follows immediately since a manifold is locally Euclidean.

Proposition 19.5. *Any manifold is normal.*

Proof. This follows from Theorem 18.16. To see that a manifold M is locally compact, consider a point $x\in M$. Then x has a Euclidean neighborhood $x\in U\subseteq M$. U is homeomorphic to an open subset V of $\mathbb{R}^n$, so we can find a compact neighborhood K of x in V (think of a closed ball in $\mathbb{R}^n$). Under the homeomorphism, K corresponds to a compact neighborhood of x in U. ■

It also follows similarly that any manifold is metrizable, but we can do better. It is convenient to introduce the following term.

19.2. Embedding.

Theorem 19.6. *Any manifold M^n admits an embedding into some Euclidean space $\mathbb{R}^N$.*

Sketch. The theorem is true as stated, but we only prove it in the case of a compact manifold. Note that in this case, since M is compact and $\mathbb{R}^N$ is Hausdorff, it is enough to find a continuous injection of M into some $\mathbb{R}^N$.

Since M is a manifold, it has an open cover by sets that are homeomorphic to $\mathbb{R}^n$. Since it is compact, there is a finite subcover $\{U_1,\dots,U_k\}$. The idea is to then use Urysohn's lemma to extend these homeomorphisms $U_i\cong\mathbb{R}^n$ to functions $f_i: M\longrightarrow\mathbb{R}^n$. Technically, this uses what is called a "partition of unity". Then the collection of functions $\{f_i\}$ give a single function $f: M\longrightarrow(\mathbb{R}^n)^k$. Often, this is an injection, but if the cover is not very well-behaved then it is necessary to also tack on the k Urysohn functions in order to get an injection $M\hookrightarrow\mathbb{R}^{nk+k}$. ■

In fact, one can do better. Munkres shows (Cor. 50.8) that every compact n-manifold embeds inside $\mathbb{R}^{2n+1}$.

Wed, Nov. 15

20. Mapping Spaces

The last main topic from the introductory part of the course on metric spaces is the idea of a function space. Given any two spaces A and Y, we will want to be able to define a topology on the set of continuous functions $A \longrightarrow Y$ in a sensible way. We already know one topology on Y^A, namely the product topology. But this does not use the topology on A at all.

Let's forget about topology for a second. Recall from the beginning of the course that a function $h : X \times A \longrightarrow Y$ between sets is equivalent to a function

$$\Psi(h) : X \longrightarrow Y^A.$$

Given h, the map $\Psi(h)$ is defined by $\big(\Psi(h)(x)\big)(a) = h(x,a)$. Conversely, given $\Psi(h)$, the function h can be recovered by the same formula.

Let's play the same game in topology. What we want to say is that a continuous map $h : X \times A \longrightarrow Y$ is the same as a continuous map $X \longrightarrow \mathrm{Map}(A,Y)$, for some appropriate *space* of maps $\mathrm{Map}(A,Y)$. Let's start by seeing why the product topology does *not* have this property.

We write $\mathcal{C}(X,Z)$ for the *set* of continuous maps $X \longrightarrow Z$. It is not difficult to check that the set-theoretic construction from above does give a function

$$\mathcal{C}(X \times A, Y) \longrightarrow \mathcal{C}(X, Y^A),$$

where for the moment Y^A denotes the set of continuous functions $A \longrightarrow Y$ given the product topology. But this function is not surjective.

Example 20.1. Take $A = [0,1]$, $Y = \mathbb{R}$, and $X = Y^A = \mathbb{R}^{[0,1]}$. We can consider the identity map $\mathbb{R}^{[0,1]} \longrightarrow \mathbb{R}^{[0,1]}$. We would like this to correspond to a continuous map $\mathbb{R}^{[0,1]} \times [0,1] \longrightarrow \mathbb{R}$. We see that, ignoring the topology, this function must be the evaluation function $ev : (g,x) \mapsto g(x)$. But this is not continuous.

To see this consider $ev^{-1}((0,1))$. If we denote by $\iota : [0,1] \hookrightarrow \mathbb{R}$ the inclusion, then the point $(\iota, 1/2)$ lies in this preimage, but we claim that no neighborhood of this point is contained in the preimage. In fact, we claim no basic neighborhood $U \times (a,b)$ lies in the preimage. For such a U must consist of functions that are close to $\iota : [0,1] \longrightarrow \mathbb{R}$ at *finitely many points* $c_1, \dots, c_n$. So given any such U and any interval $(a,b) = (1/2 - \epsilon, 1/2 + \epsilon)$, pick any point $d \in (a,b)$ that is distinct from the c_i. Then construct a continuous function $g : [0,1] \longrightarrow \mathbb{R}$ such that

(1) $g(c_i) = c_i$ for each i and
(2) $g(d) =$ two bajillion.

Then $(g,d) \in U \times (a,b)$ but $(g,d) \notin ev^{-1}((0,1))$ since $ev(g,d) = g(d) =$two bajillion.

The **compact-open** topology on the set $\mathcal{C}(A,Y)$ has a prebasis given by

$$S(K,U) = \{f : A \longrightarrow Y \mid f(K) \subseteq U\},$$

where K is compact and $U \subseteq Y$ is open. We write $\mathrm{Map}(A,Y)$ for the set $\mathcal{C}(A,Y)$ equipped with the compact-open topology.

Theorem 20.2. *Suppose that A is locally compact Hausdorff. Then a function $f : X \times A \longrightarrow Y$ is continuous if and only if the induced function $g = \Psi(f) : X \longrightarrow \mathrm{Map}(A,Y)$ is continuous.*

Proof. ($\Rightarrow$) This direction does not need that A is locally compact. Before we give the proof, we should note why $\Psi(f)(x) : A \longrightarrow Y$ is continuous. This map is the composite $A \xrightarrow{\iota_x} X \times A \xrightarrow{f} Y$ and therefore continuous.

We now wish to show that $g = \Psi(f)$ is continuous. Let $S(K,U)$ be a sub-basis element in $\mathrm{Map}(A,Y)$. We wish to show that $g^{-1}(S(K,U))$ is open in X. Let $g(x) = f(x,-) \in S(K,U)$.

Since f is continuous, the preimage $f^{-1}(U) \subseteq X \times A$ is open. Furthermore, $\{x\} \times K \subseteq f^{-1}(U)$. We wish to use the Tube Lemma, so we restrict from $X \times A$ to $X \times K$. By the Tube Lemma, we can find a basic neighborhood V of x such that $V \times K \subseteq (X \times K) \cap f^{-1}(U)$. It follows that $g(V) \subseteq S(K, U)$, so that V is a neighborhood of x in $g^{-1}(S(K, U))$.

Fri, Nov. 17 ($\Leftarrow$) Suppose that g is continuous. Note that we can write f as the composition

$$X \times A \xrightarrow{g \times \mathrm{id}} \mathrm{Map}(A, Y) \times A \xrightarrow{ev} Y,$$

so it is enough to show that ev is continuous.

Lemma 20.3. *The map $ev : \mathrm{Map}(A, Y) \times A \longrightarrow Y$ is continuous if A is locally compact Hausdorff.*

Proof. Let $U \subseteq Y$ be open and take a point (f, a) in $ev^{-1}(U)$. This means that $f(a) \in U$. Since A is locally compact Hausdorff, by Homework 9.3 we can find a compact neighborhood K of a contained in $f^{-1}(U)$ (this is open since f is continuous). It follows that $S(K, U)$ is a neighborhood of f in $\mathrm{Map}(A, Y)$, so that $S(K, U) \times K$ is a neighborhood of (f, a) in $ev^{-1}(U)$. ∎

∎

20.1. **Hom-Tensor Adjunction.** Even better, we have

Theorem 20.4. *Let X and A be locally compact Hausdorff. Then the above maps give homeomorphisms*

$$\mathrm{Map}(X \times A, Y) \cong \mathrm{Map}(X, \mathrm{Map}(A, Y)).$$

It is fairly simple to construct a continuous map in either direction, using Theorem 20.2. You should convince yourself that the two maps produced are in fact inverse to each other.

In practice, it's a bit annoying to keep track of these extra hypotheses at all times, especially since not all constructions will preserve these properties. It turns out that there is a "convenient" category of spaces, where everything works nicely.

Definition 20.5. A space A is **compactly generated** if a subset $B \subseteq A$ is closed if and only if for every map $u : K \longrightarrow A$, where K is compact Hausdorff, then $u^{-1}(B) \subseteq K$ is closed.

We say that the topology of A is determined (or generated) by compact subsets. Examples of compactly generated spaces include locally compact spaces and first countable spaces.

Definition 20.6. A space X is **weak Hausdorff** if the image of every $u : K \longrightarrow X$ is closed in X.

There is a way to turn any space into a weak Hausdorff compactly generated space. In that land, everything works well! For the most part, whenever an algebraic topologist says "space", they really mean a compactly generated weak Hausdorff space. Next semester, we will always implicity be working with spaces that are CGWH.

Looking back to the initial discussion of metric spaces, there we introduced the uniform topology on a mapping space.

Theorem 20.7 (Munkres, 46.7 or Willard, 43.6)**.** *Let Y be a metric space. Then on the set $\mathcal{C}(A, Y)$ of continuous functions $A \longrightarrow Y$, the compact-open topology is intermediate between the uniform topology and the product topology. Furthermore, the compact-open topology agrees with the uniform topology if A is compact.*

21. CW complexes

Recently, we consider topological manifolds, which are a nice collection of spaces. Next semester, you will often work with another nice collection of spaces that can be built inductively. These are cell complexes, or CW complexes.

A typical example is a sphere. In dimension 1, we have S^1, which we can represent as the quotient of $I = [0, 1]$ by endpoint identification. Another way to say this is that we start with a point, and we "attach" an interval to that point by gluing both ends to the given point.

For S^2, there are several possibilities. One is to start with a point and glue a disk to the point (glueing the boundary to the point). An alternative is to start with a point, then attach an interval to get a circle. To this circle, we can attach a disk, but this just gives us a disk again, which we think of as a hemisphere. If we then attach a second disk (the other hemisphere), we get S^2.

But what do we really mean by "attach a disk"?

21.1. Pushouts.

Let's start today by discussing the general "pushout" construction.

Definition 21.1. Suppose that $f : A \longrightarrow X$ and $g : A \longrightarrow Y$ are continuous maps. The **pushout** (or glueing construction) of X and Y along A is defined as

$$X \cup_A Y := X \amalg Y/\sim, \qquad f(a) \sim g(a).$$

We have an inclusion $X \hookrightarrow X \amalg Y$. Composing this with the quotient map to $X \cup_A Y$ gives the map $\iota_X : X \longrightarrow X \cup_A Y$. We similarly have a map $\iota_Y : Y \longrightarrow X \cup_A Y$. Moreover, these maps make the diagram to the right commute. The point is that

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\iota_Y"] \\ X \arrow[r, "\iota_X"] & X \cup_A Y \end{tikzcd}$$

$$\iota_X(f(a)) = \overline{f(a)} = \overline{g(a)} = \iota_Y(g(a)).$$

The main point of this construction is the following property.

Proposition 21.2 (Universal property of the pushout)**.** *Suppose that $\varphi_1 : X \longrightarrow Z$ and $\varphi_2 : Y \longrightarrow Z$ are maps such that $\varphi_1 \circ f = \varphi_2 \circ g$. Then there is a unique map $\Phi : X \cup_A Y \longrightarrow Z$ which makes the diagram to the right commute.*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\iota_Y"] \arrow[ddr, bend left, "\varphi_2"] & \\ X \arrow[r, "\iota_X"] \arrow[drr, bend right, "\varphi_1"'] & X \cup_A Y \arrow[dr, dashed, "\Phi"] & \\ & & Z \end{tikzcd}$$

This generalizes the "pasting" lemma. Suppose that $U, V \subseteq X$ are open subsets with $X = U \cup V$. Then it is not difficult to show that the pushout $U \cup_{U \cap V} V$ is homeomorphic to X. The universal property for the pushout then says that specifying a continuous map out of X is the same as specifying a pair of continuous maps out of U and V which agree on their intersection $U \cap V$. This is precisely the statement of the pasting lemma!

Mon, Nov. 20

Definition 21.3. (Attaching an interval) Given a space X and two points $x \neq y \in X$, we get a continuous map $\alpha : S^0 \longrightarrow X$ with $\alpha(0) = x$ and $\alpha(1) = y$. There is the standard inclusion $S^0 \hookrightarrow D^1 = [-1, 1]$, and we write $X \cup_\alpha D^1$ for the pushout

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d, "\alpha"'] & D^1 \arrow[d] \\ X \arrow[r, "\iota_X"] & X \cup_\alpha D^1 \end{tikzcd}$$

The image $\iota(\mathrm{Int}(D^1))$ is referred to as a 1-cell and is sometimes denoted e^1. Thus the above space, which is described as obtained by attaching an 1-cell to X, is also written $X \cup_\alpha \overline{e^1}$ or $X \cup_\alpha e^1$.

Generalizing the construction from last time, for any n, we have the standard inclusion $S^{n-1} \hookrightarrow D^n$ as the boundary.

Definition 21.4. Given a space X and a continuous map $\alpha : S^{n-1} \longrightarrow X$, we write $X \cup_\alpha D^n$ for the pushout

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, "\alpha"'] & D^n \arrow[d] \\ X \arrow[r, "\iota_X"] & X \cup_\alpha D^n \end{tikzcd}$$

The image $\iota(\mathrm{Int}(D^n))$ is referred to as an n-cell and is sometimes denoted e^n. Thus the above space, which is described as obtained by attaching an n-cell to X, is also written $X \cup_\alpha \overline{e^n}$ or $X \cup_\alpha e^n$.

In general, this attaching process does not disturb the interiors of the cells, as follows from the following, which you are asked to show on homework.

Proposition 21.5. *If $g : A \hookrightarrow Y$ is injective, then $\iota_X : X \longrightarrow X \cup_A Y$ is also injective.*

Example 21.6. If $A = \emptyset$, then $X \cup_A Y = X \amalg Y$.

Example 21.7. If $A = *$, then $X \cup_A Y = X \vee Y$.

Example 21.8. If $A \subseteq X$ is a subspace and $Y = *$, then $X \cup_A * \cong X/A$.

By the way, Proposition 21.5 is not only true for injections.

Proposition 21.9. *(i) If $f : A \longrightarrow X$ is surjective, then so is $\iota_Y : Y \longrightarrow X \cup_A Y$.*
(ii) If $f : A \longrightarrow X$ is a homeomorphism, then so is $\iota_Y : Y \longrightarrow X \cup_A Y$.

Proof. We prove only (ii). We show that if f is a homeomorphism, then Y satisfies the same universal property as the pushout. Consider the test diagram to the right. We have no choice but to set $\Phi = \varphi_2$. Does this make the diagram commute? We need to check that $\Phi \circ g \circ f^{-1} = \varphi_1$. Well,

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[d, "f"'] & & Y \arrow[d, equal] \arrow[ddr, bend left, "\varphi_2"] & \\ X \arrow[r, "f^{-1}"] \arrow[rrrd, bend right, "\varphi_1"'] & A \arrow[r, "g"] & Y \arrow[dr, dashed, "\Phi"] & \\ & & & Z \end{tikzcd}$$

$$\Phi \circ g \circ f^{-1} = \varphi_2 \circ g \circ f^{-1} = \varphi_1 \circ f \circ f^{-1} = \varphi_1.$$

■

21.2. **Cell complexes.** We use the idea of attaching cells (using a pushout) to inductively build up the idea of a cell complex or CW complex.

Definition 21.10. A **CW complex** is a space built in the following way

(1) Start with a discrete set X^0 (called the set of 0-cells, or the 0-skeleton)
(2) Given the $(n-1)$-skeleton X^{n-1}, the n-skeleton X^n is obtained by attaching n-cells to X^{n-1}.
(3) The space X is the union of the X^n, topologized using the "weak topology". This means that $U \subseteq X$ is open if and only if $U \cap X^n$ is open for all n.

The third condition is not needed if $X = X^n$ for some n (so that X has no cells in higher dimensions). On the other hand, the 'W' in the name CW complex refers to item 3 ("weak topology"). The 'C' in CW complex refers to the **C**losure finite property: the closure of any cell is contained in a finite union of cells. We will come back to this point later.

According to condition (2), the n-skeleton is obtained from the $(n-1)$-skeleton by attaching cells. Often, we think of this as attaching one cell at a time, but we can equally well attach them all at once, yielding a pushout diagram

$$\begin{tikzcd} \coprod_{\mathcal{E}_n} S^{n-1} \arrow[r] \arrow[d] & \coprod_{\mathcal{E}_n} D^n \arrow[d] \\ X^{n-1} \arrow[r] & X^n \end{tikzcd}$$

for each n. The maps $S^{n-1} \longrightarrow X^{n-1}$ are called the **attaching maps** for the cells, and the resulting maps $D^n \longrightarrow X^n$ are called the **characteristic maps**.

Example 21.11. (1) S^n. We have already discussed two CW structures on S^2. The first has X^0 a singleton and a single n-cell attached. The other had a single 0-cell and single 1-cell but two 2-cells attached. There is a third option, which is to start with two 0-cells, attach two 1-cells to get a circle, and then attach two 2-cells to get S^2.

The first and third CW structures generalize to any S^n. There is a minimal CW structure having a single 0-cell and single n-cell, and there is another CW structure have two cells in every dimension up to n.

Mon, Nov. 27

Last time, we were discussing CW complexes, and we considered two different CW structures on S^n. We continue with more examples.

(2) (Torus) In general, a product of two CW complexes becomes a CW complex. We will describe this in the case $S^1 \times S^1$, where S^1 is built using a single 0-cell and single 1-cell.

Start with a single 0-cell, and attach two 1-cells. This gives $S^1 \vee S^1$. Now attach a single 2-cell to the 1-skeleton via the attaching map ψ defined as follows. Let us refer to the two circles in $S^1 \vee S^1$ as ℓ and r. We then specify $\psi : S^1 \longrightarrow S^1 \vee S^1$ by $\ell r \ell^{-1} r^{-1}$. What we mean is to trace out ℓ on the first quarter of the domain, to trace out r on the second quarter, to run ℓ in reverse on the third quarter, and finally to run r in reverse on the final quarter.

We claim that the resulting CW complex X is the torus. Since the attaching map $\psi : S^1 \longrightarrow S^1 \vee S^1$ is surjective, so is $\iota_{D^2} : D^2 \longrightarrow X$. Even better, it is a quotient map. On the other hand, we also have a quotient map $I^2 \longrightarrow T^2$, and using the homeomorphism $I^2 \cong D^2$ from before, we can see that the quotient relation in the two cases agrees. We say that this homeomorphism $T^2 \cong X$ puts a cell structure on the torus. There is a single 0-cell (a vertex), two 1-cells (the two circles in $S^1 \vee S^1$), and a single 2-cell.

(3) $\mathbb{RP}^n$. Let's start with $\mathbb{RP}^2$. Recall that one model for this space was as the quotient of D^2, where we imposed the relation $x \sim -x$ on the boundary. If we restrict our attention to the boundary S^1, then the resulting quotient is $\mathbb{RP}^1$, which is again a circle. The quotient map $q : S^1 \longrightarrow S^1$ is the map that winds twice around the circle. In complex coordinates, this would be $z \mapsto z^2$. The above says that we can represent $\mathbb{RP}^2$ as the pushout

$$\begin{tikzcd} S^1 \arrow[r, "\iota"] \arrow[d, "q"'] & D^2 \arrow[d] \\ S^1 \arrow[r] & \mathbb{RP}^2 \end{tikzcd}$$

If we build the 1-skeleton S^1 using a single 0-cell and a single 1-cell, then $\mathbb{RP}^2$ has a single cell in dimensions ≤ 2.

More generally, we can define $\mathbb{RP}^n$ as a quotient of D^n by the relation $x \sim -x$ on the boundary S^{n-1}. This quotient space of the boundary was our original definition of $\mathbb{RP}^{n-1}$. It follows that we can describe $\mathbb{RP}^n$ as the pushout

$$\begin{tikzcd} S^{n-1} \arrow[r, "\iota"] \arrow[d, "q"'] & D^n \arrow[d] \\ \mathbb{RP}^{n-1} \arrow[r] & \mathbb{RP}^n \end{tikzcd}$$

Thus $\mathbb{RP}^n$ can be built as a CW complex with a single cell in each dimension $\leq n$.

(4) $\mathbb{CP}^n$. Recall that $\mathbb{CP}^1 \cong S^2$. We can think of this as having a single 0-cell and a single 2-cell. We defined $\mathbb{CP}^2$ as the quotient of S^3 by an action of S^1 (thought of as $U(1)$). Let $\eta : S^3 \longrightarrow \mathbb{CP}^1$ be the quotient map. What space do we get by attaching a 4-cell to $\mathbb{CP}^1$ by the map η? Well, the map η is a quotient, so the pushout $\mathbb{CP}^1 \cup_\eta D^4$ is a quotient of D^4 by the S^1-action on the boundary.

Wed, Nov. 29

Now include D^4 into $S^5 \subseteq \mathbb{C}^3$ via the map

$$\varphi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4, \sqrt{1 - \sum x_i^2}, 0).$$

(This would be a hemi-equator.) We have the diagonal $U(1)$ action on S^5. But since any nonzero complex number can be rotated onto the positive x-axis, the image of φ meets every S^1-orbit in S^5, and this inclusion induces a homeomorphism on orbit spaces

$$D^4/U(1) \quad \cong S^5/U(1) = \mathbb{CP}^2.$$

We have shown that $\mathbb{CP}^2$ has a cell structure with a single 0-cell, 2-cell, and 4-cell.

This story of course generalizes to show that any $\mathbb{CP}^n$ can be built as a CW complex having a cell in each even dimension.

Let's talk about some of the (nice!) topological properties of CW complexes.

21.3. **Niceness.**

Theorem 21.12 (Hatcher, Prop A.3)**.** *Any CW complex X is normal.*

Even better,

Theorem 21.13 (Lee, Theorem 5.22)**.** *Every CW complex is paracompact.*

Proposition 21.14. *Any CW complex X is locally path-connected.*

Proof. Let $x \in X$ and let U be any open neighborhood of x. We want to find a path-connected neighborhood V of x in U. Recall that a subset $V \subseteq X$ is open if and only if $V \cap X^n$ is open for all n. We will define V by specifying open subsets $V^n \subseteq X^n$ with $V^{n+1} \cap X^n = V^n$ and then setting $V = \cup V^n$.

Suppose that x is contained in the (interior of the) cell e_i^n. We set $V^k = \emptyset$ for $k < n$. We specify V_n by defining $\Phi_j^{-1}(V^n)$ for each n-cell e_j^n. If $j \neq i$, we set $\Phi_j^{-1}(V_n) = \emptyset$. We define $\Phi_i^{-1}(V_n)$ to be an open n-disc around $\Phi_i^{-1}(x)$ whose closure is contained in $\Phi_i^{-1}(U)$. Now suppose we have defined V^k for some $k \geq n$. Again, we define V^{k+1} by defining each $\Phi_j^{-1}(V^{k+1})$. By assumption, $\overline{\Phi_j^{-1}(V^k)} \subseteq \partial D^{k+1} \subseteq \Phi_j^{-1}(U)$. By the Tube lemma, there is an $\epsilon > 0$ such that (using radial coordinates) $\Phi_j^{-1}(V^k) \times (1-\epsilon, 1] \subset \Phi_j^{-1}(U)$. We define

$$\Phi_j^{-1}(V^{k+1}) = \Phi_j^{-1}(V^k) \times [1, 1-\epsilon/2),$$

which is path-connected by induction. Note that this forces $\Phi_j^{-1}(V^{k+1})$ to be empty if the image of the attaching map for the cell e_j^{k+1} does not meet V_k. Now by construction V^{k+1} is the overlapping union of path-connected sets and therefore path-connected. This also guarantees that $\overline{V^{k+1}} \subset U \cap X^{k+1}$, allowing the induction to proceed. ■

Proposition 21.15 (Hatcher, A.1)**.** *Any compact subset K of a CW complex X meets finitely many cells.*

Corollary 21.16. *Any CW complex has the closure-finite property, meaning that the closure of any cell meets finitely many cells.*

Proof. The closure of e_i is $\Phi_i(D_i^{n_i})$, which is compact. The result follows from the proposition. ■

Corollary 21.17.

(i) A CW complex X is compact if and only if it has finitely many cells.

(ii) A CW complex X is locally compact if and only if the collection $\mathcal{E}$ of cells is locally finite.

Part 6. Homotopy and the fundamental group

22. Homotopy

Fri, Dec. 1

We have studied a number of topological properties of spaces, but how would we use these to distinguish S^2, $\mathbb{RP}^2$, and T^2? These are all compact, connected 2-manifolds. It turns out that the fundamental group will allow us to distinguish these spaces. This is the start of **algebraic** topology. We first introduce the idea of a homotopy.

Definition 22.1. Given maps f and $g : X \longrightarrow Y$, a **homotopy** h between f and g is a map $h : X \times I \longrightarrow Y$ ($I = [0,1]$) such that $f(x) = h(x,0)$ and $g(x) = h(x,1)$. We say f and g are **homotopic** if there exists a homotopy between them (and write $h : f \simeq g$).

Example 22.2. Let $f = \mathrm{id} : \mathbb{R} \longrightarrow \mathbb{R}$ and take $g : \mathbb{R} \longrightarrow \mathbb{R}$ to be the constant map $g(x) = 0$. Then a homotopy $h : f \simeq g$ is given by

$$h(x,t) = x(1-t).$$

Check that $h(x,0) = f(x)$ and $h(x,1) = g(x)$. Since f is homotopic to a constant map, we say that f is **null-homotopic** (and h is a **null-homotopy**).

Example 22.3. Consider $f = \mathrm{id} : S^1 \longrightarrow S^1$ and the map $g : S^1 \longrightarrow S^1$ defined by $g(\cos(\theta), \sin(\theta)) = (\cos(2\theta), \sin(2\theta))$. Thinking of S^1 as the complex numbers of unit norm, the map g can alternatively be described as $g(z) = z^2$. Then the maps f and g are <u>not</u> homotopic. Furthermore, neither is null-homotopic. (Though we won't be able to show this until next semester.)

Proposition 22.4. *The property of being homotopic defines an equivalence relation on the set of maps $X \longrightarrow Y$.*

Proof. (Reflexive): Need to show $f \simeq f$. Use the **constant homotopy** defined by $h(x,t) = f(x)$ for all t.

(Symmetric): If $h : f \simeq g$, we need a homotopy from g to f. Define $H(x,t) = h(x, 1-t)$ (reverse time).

(Transitive): If $h_1 : f_1 \simeq f_2$ and $h_2 : f_2 \simeq f_3$, we define a new homotopy h from f_1 to f_3 by the formula

$$h(x,t) = \begin{cases} h_1(x, 2t) & 0 \le t \le 1/2 \\ h_2(x, 2t-1) & 1/2 \le t \le 2. \end{cases}$$

■

We write $[X, Y]$ for the set of homotopy classes of maps $X \longrightarrow Y$.

Proposition 22.5. *(Interaction of composition and homotopy) Suppose given maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $X \xrightarrow{f'} Y \xrightarrow{g'} Z$. If $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f'$.*

Proof. We will show that $g \circ f \simeq g' \circ f$. The required homotopy is given by

$$H(x,t) = h'(f(x), t).$$

It is easily verified that $H(x,0) = g \circ f(x)$ and $H(x,1) = g' \circ f(x)$. Why is the map $H : X \times I \longrightarrow Z$ continuous? It is the composition of the continuous maps

$$X \times I \xrightarrow{f \times \mathrm{id}} Y \times I \xrightarrow{h'} Z.$$

That the map $f \times \mathrm{id}$ is continuous can be easily verified using the universal property. ■

Definition 22.6. A map $f : X \longrightarrow Y$ is a **homotopy equivalence** if there is a map $g : Y \longrightarrow X$ such that both composites $f \circ g$ and $g \circ f$ are homotopic to the identity maps. We say that spaces X and Y are **homotopy equivalent** if there exists some homotopy equivalence between them, and we write $X \simeq Y$.

Remark 22.7. It is clear that any homeomorphism is a homotopy equivalence, since then both composites are *equal* to the idenitity maps.

The following example shows that the converse is not true.

Example 22.8. The (unique) map $f : \mathbb{R} \longrightarrow *$, where $*$ is the one-point space, is a homotopy equivalence. Pick *any* map $g : * \longrightarrow \mathbb{R}$ (for example, the inclusion of the origin). Then $f \circ g = \mathrm{id}$. The other composition $g \circ f : \mathbb{R} \longrightarrow \mathbb{R}$ is contant, but we have already seen last time that the identity map of $\mathbb{R}$ is null-homotopic. So $\mathbb{R} \simeq *$. The same argument works equally well to show that $\mathbb{R}^n \simeq *$ for any n. Even more generally, if X is a convex subset of $\mathbb{R}^n$, then $X \simeq *$.

Here's some more terminology: any space that is homotopy-equivalent to the one-point space is said to be **contractible**. As we have just seen in the example above, this is equivalent to the statement that the identity map is null-homotopic.

More generally, we can show that any two maps $f, g : X \rightrightarrows \mathbb{R}^n$ are homotopic. The **straight-line homotopy** between f and g is given by

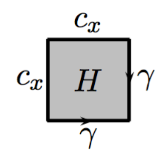
$$h(x, t) = (1 - t)f(x) + tg(t).$$

We will see next semester that the spaces S^2, $\mathbb{RP}^2$, and T^2 are not homotopy-equivalent (and therefore not homeomorphic).

22.1. **Path-homotopy.**

Mon, Dec. 4

Recall that a **path** in a space X is simply a continuous map $\gamma : I \longrightarrow X$. It will turn out to be fruitful to study homotopy-classes of paths in a space X. But this is not very interesting if we don't impose additional restrictions: every path is null! A contracting homotopy for the path γ is given by

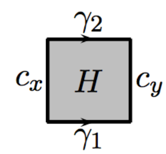


$$H(s,t) = \gamma(s(1-t)).$$

We need to modify our notion of homotopy to get an interesting relation for paths.

Definition 22.9. Let γ_1 and γ_2 be paths in X with the same initial and end points. A **path-homotopy** between γ_1 and γ_2 is simply a homotopy h such that at each time t, the resulting path $h(-,t)$ also has the same initial and end points as γ_1 and γ_2.

Another way to think about this is that a path homotopy is a map from the square $I \times I$ that is constant on the left vertical edge and also on the right vertical edge.



Example 22.10. The two paths $\gamma_1(s) = e^{i\pi s}$ and $\gamma_2(s) = e^{-i\pi s}$ are path-homotopic in $\mathbb{R}^2$. A homotopy is given by $h(s,t) = (1-t)\gamma_1(s) + t\gamma_2(s)$. This is the **straight-line homotopy**. For example, when we restrict to $s = 1/2$, the homotopy gives the vertical diameter of the circle.

On the other hand, we could also consider these as paths in $\mathbb{R}^2 - \{(0,0)\}$ or as paths in S^1. We will see later that these are **not** path-homotopic in either of these spaces.

Proposition 22.11. *Given two points a and b in X, path-homotopy defines an equivalence relation on the set of paths from a to b.*

A path in X that begins and ends at the <u>same</u> point is called a **loop** in X. We call the starting/end point the **basepoint** of the loop (and often of X as well). By the above proposition, path-homotopy defines an equivalence relation on the set of loops in X with basepoint x_0. The set of equivalence classes is denoted $\pi_1(X, x_0)$ and is called the **fundamental group of** X (with basepoint x_0). Of course, so far we have no reason to call this a group, we only know this as a set.

Example 22.12. Use of straight-line homotopies show that $\pi_1(\mathbb{R}^n, x) = \{c_x\}$ for any n and x. More generally, $\pi_1(X, x) = \{c_x\}$ for any convex subset of $\mathbb{R}^n$. This holds even more generally for any **star-shaped** region in $\mathbb{R}^n$. A subset $X \subset \mathbb{R}^n$ is said to be star-shaped around x if for any $y \in X$, the straight-line segment $\overline{xy}$ is contained in X.

Here is a slightly different perspective on loops. Since a loop is a map $\gamma : I \longrightarrow X$ that is constant on the subspace $\partial I = \{0,1\} \subseteq I$, there is an induced map from the quotient space $\overline{\gamma} : I/\partial I \longrightarrow X$. Recall that $I/\partial I$ is homeomorphic to the circle S^1. So a loop in X is the same as a map $\overline{\gamma} : S^1 \longrightarrow X$.

A **based map** between two spaces with chosen basepoints is simply a map that takes the basepoint of one space to the basepoint of the other. By a **based homotopy**, we mean a homotopy through based maps (so the homotopy is constant on the basepoint). Based homotopy defines an equivalence relation on the set of based maps, and the set of based homotopy classes is denoted

$$[(X, x_0), (Y, y_0)]_*.$$

It is customary to take $(1,0)$ as the basepoint for S^1, and path-homotopy classes of loops in X, based at x_0, correspond to based homotopy classes of maps $S^1 \longrightarrow X$. So

$$\pi_1(X, x_0) \cong [(S^1, (1,0)), (X, x_0)]_*.$$

Where does the group structure on homotopy classes of loops come from? Well, you can concatenate paths, by traveling first along one and then along the other.

Definition 22.13. Let γ and λ be paths in X. We say the two paths are **composable** in X if $\gamma(1) = \lambda(0)$. When this is the case, we define the **concatenation** of γ and λ to be the path

$$\gamma \cdot \lambda(s) = \begin{cases} \gamma(2s) & s \in [0, 1/2] \\ \lambda(2s-1) & s \in [1/2, 1]. \end{cases}$$

This formula looks familiar, right? This was the one used in Proposition 22.4 to glue two homotopies together. This is no accident: a path is precisely a homotopy between two constant maps!

Remark 22.14. Beware that $\gamma \cdot \lambda$ means do γ first (in double time), and then λ (in double time). This is the opposite convention of what we use for function composition.

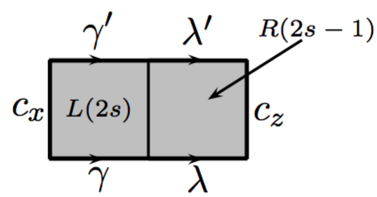
Concatenation will provide the group structure on $\pi_1(X)$.

Proposition 22.15. *The above operation only depends on path-homotopy classes. That is, if $\gamma \simeq_p \gamma'$ and $\lambda \simeq_p \lambda'$, then $\gamma \cdot \lambda \simeq_p \gamma' \cdot \lambda'$.*

Proof. Let $L : \gamma \simeq_p \gamma'$ and $R : \lambda \simeq_p \lambda'$ be path-homotopies. We define a new path homotopy by

$$H(s,t) = \begin{cases} L(2s, t) & s \in [0, 1/2] \\ R(2s-1, t) & s \in [1/2, 1]. \end{cases}$$

■



This tells us that the concatenation operation is well-defined on path-homotopy classes. We will next check that it gives a well-behaved algebraic operation.

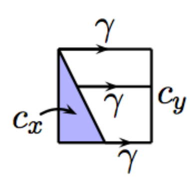
22.2. **The fundamental group.** For any point $x \in X$, we denote by c_x the **constant path** at x in X.

Proposition 22.16. *Let γ (from x to y), λ, and μ be composable paths in X. Concatenation of path-homotopy classes satisfies the following properties.*

1. *(unit law)* $[c_x] \cdot [\gamma] = [\gamma] = [\gamma] \cdot [c_y]$
2. *(associativity)* $([\gamma] \cdot [\lambda]) \cdot [\mu] = [\gamma] \cdot ([\lambda] \cdot [\mu])$
3. *(inverses) Define $\overline{\gamma}(s) = \gamma(1-s)$. Then $[\gamma] \cdot [\overline{\gamma}] = [c_x]$ and $[\overline{\gamma}] \cdot [\gamma] = [c_y]$.*

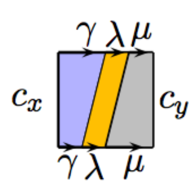
Proof. (1) Define

$$h(s,t) = \begin{cases} x & 2s \in [0, 1-t] \\ \gamma(\frac{2s-1+t}{1+t}) & 2s \in [1-t, 2]. \end{cases}$$



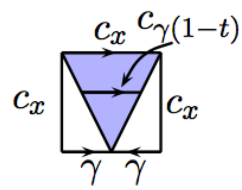
(2) Define

$$h(s,t) = \begin{cases} \gamma(\frac{4s}{1+t}) & s \in [0, \frac{1+t}{4}] \\ \lambda(4s-1-t) & s \in [\frac{1+t}{4}, \frac{2+t}{4}] \\ \mu(\frac{4s-2-t}{2-t}) & s \in [\frac{2+t}{4}, 1]. \end{cases}$$



(3) Define

$$h(s,t) = \begin{cases} \gamma(2s) & 2s \in [0, 1-t] \\ \gamma(1-t) & 2s \in [1-t, 1+t] \\ \gamma(2(1-s)) & 2s \in [1+t, 2]. \end{cases}$$



Actually, for parts (1) and (2) there is a slicker approach, (this is in Hatcher). A **reparametrization** of a path γ is a composition $\gamma \circ \varphi$, where $\varphi : I \longrightarrow I$ is any map satisfying $\varphi(0) = 0$ and $\varphi(1) = 1$. It is clear that any such φ is homotopic to the identity map of I (just use a straight-line homotopy). For (1), we can write $c_x \cdot \gamma$ as a reparametrization of γ. Thus $c_x \cdot \gamma = \gamma \circ \varphi \simeq_p \gamma$. A similar argument also works for (2). ∎

$\boxed{\textbf{Wed, Dec. 6}}$

Ok, now we know that we have a group structure on $\pi_1(X, x_0)$! Next semester, we will show the following result:

Theorem 22.17. *The fundamental group $\pi_1(S^1, 1)$ is an infinite cyclic group. In other words, it is isomorphic to $\mathbb{Z}$.*

It is easy to write down a group homomorphism $\mathbb{Z} \xrightarrow{\phi} \pi_1(S^1, 1)$. We define $\phi(n)$ to be the loop that winds around the circle n times. In other words,

$$\phi(n)(t) = e^{t \cdot 2n\pi i}.$$

The content of the theorem is that this homomorphism is bijective.

We can derive a number of very interesting consequences from our knowledge of the fundamental group of S^1.

First, we discuss how the fundamental group interacts with maps.

Proposition 22.18. *Let $(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$ be based maps.*

1. *The induced map $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is a group homomorphism.*
2. *The composition $g_* \circ f_*$ agrees with $(g \circ f)_*$.*
3. *If $(Y, y_0) = (X, x_0)$ and $f \simeq_p \mathrm{id}$, then $f_* = \mathrm{id}$.*

Taken together these statements say that the assignment $(X, x_0) \mapsto \pi_1(X, x_0)$ defines a **functor** from the category of based spaces and basepoint-preserving maps to the category of groups and group homomorphisms.

Theorem 22.19. *(Brouwer fixed point theorem) For any map $f : D^2 \longrightarrow D^2$, there exists at least one point $x \in D^2$ such that $f(x) = x$. Such an x is called a* **fixed point** *of the map f.*

Proof. Assume for a contradiction that f has *no* fixed points. Then $x - f(x)$ is not the origin, and for each point x there is a unique $t_x \geq 1$ such that $f(x) + t_x(x - f(x))$ lies on S^1. This is where the ray starting at $f(x)$ and passing through x meets the circle. Define $g(x) : D^2 \longrightarrow S^1$ by the formula

$$g(x) = f(x) + t_x(x - f(x)).$$

You should convince yourself that t_x, and therefore $g(x)$, is a continuous function of x.

Now the key point is that if x starts in the boundary S^1 of D^2, then $t_x = 1$ and $g(x) = x$. In other words, the composition

$$S^1 \xrightarrow{i} D^2 \xrightarrow{g} S^1$$

is the identity map of S^1. Consider what happens on the fundamental group. The conclusion would be that the composition

$$\pi_1(S^1) = \mathbb{Z} \xrightarrow{i_*} \pi_1(D^2) = 0 \xrightarrow{g_*} \pi_1(S^1) = \mathbb{Z}$$

is the identity map of $\mathbb{Z}$, which is impossible. ∎

Application: Take a cup of coffee and move it around, so that the coffee gets mixed up. When it comes to rest, there is some particle that ends up where it started. (Okay, this is sort of BS since it assumes every particle stays on the surface, but it is a common description of the Brouwer fixed point theorem.)

Fri, Dec. 8

22.3. **Change of basepoint and degree.** Let $f : S^1 \longrightarrow S^1$ be any map. Then, as we saw last time, it defines a homomorphism

$$f_* : \pi_1(S^1, 1) \longrightarrow \pi_1(S^1, f(1)).$$

If f is not based, then it is a little annoying that the target fundamental group has a different choice of basepoint. There is a way to fix this. First, let γ be *any* path $\gamma : 1 \rightsquigarrow f(1)$.Then if α is any loop based at $f(1)$, we can create a loop based at 1 by the path composition

$$\Phi_\gamma(\alpha) = \gamma \cdot \alpha \cdot \gamma^{-1}.$$

The trouble is that, in general, the map Φ_γ does depend on the choice of (path-homotopy class of) γ. Any other such path is necessarily of the form $\delta = \gamma \cdot \beta$ for some loop β based at $f(1)$. Then

$$\Phi_\delta(\alpha) = \gamma \cdot \beta \cdot \alpha \cdot \beta^{-1} \cdot \gamma^{-1} = \Phi_\gamma(\beta\alpha\beta^{-1}).$$

In our case, since $\pi_1(S^1, 1)$ is **abelian**, this conjugation disappears, so that the change-of-basepoint map Φ_γ does not depend on any choice. Thus, given any (continuous) map $f : S^1 \longrightarrow S^1$, we get a well-defined homomorphism

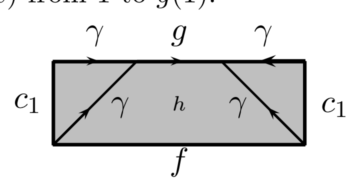
$$\mathbb{Z} \cong \pi_1(S^1, 1) \xrightarrow{f_*} \pi_1(S^1, f(1)) \xrightarrow{\Phi_\gamma} \pi_1(S^1, 1) \cong \mathbb{Z}.$$

Definition 22.20. We define the **degree** of f to be $\deg(f) = \Phi_\gamma(f_*(1))$, the image of 1 under this composition.

Proposition 22.21. *If f and g are homotopic as maps $S^1 \longrightarrow S^1$, then $\deg(f) = \deg(g)$.*

Proof. We suppose WLOG that f is based. Note that if we know that f and g are homotopic through based maps, then the result follows. The map g may not be based. Let $h : f \simeq g$ be a homotopy of maps $S^1 \longrightarrow S^1$ and let γ be the path $\gamma(t) = h(1, t)$ from 1 to $g(1)$.

If we define $\tilde{g} := \Phi_\gamma(g)$, then $\deg(\tilde{g}) = \deg(g)$ by the definition of the degree. So it suffices to identify $\deg(f)$ with $\deg(\tilde{g})$. But we can build a based homotopy as in the picture to the right.



∎

Theorem 22.22. *(Fundamental theorem of algebra) Every nonconstant polynomial with complex coefficients has a solution in $\mathbb{C}$.*

Proof. Assume that $p(z) \neq 0$ for all $z \in \mathbb{C}$. We will show that p must be constant. Define a function $f : S^1 \longrightarrow S^1$ by $f(z) = p(z)/\|p(z)\|$. We can define a homotopy by

$$h(z, t) = p(zt)/\|p(zt)\|.$$

Thus f is homotopic to a constant map, which means that it has "degree" zero.

On the other hand, write a_i for the coefficients of the degree n polynomial $p(z)$. For convenience, we assume $p(z)$ is monic. Let $k(z,t)$ be the homotopy between z^n and $p(z)$ given by the formula

$$k(z,t) = \sum_{i=0}^{n} a_i z^i t^{n-i} = z^n + a_{n-1} z^{n-1} t + \cdots + a_0 t^n.$$

Note that, for $t \neq 0$ this can be rewritten as $k(z,t) = t^n p(z/t)$. In particular, this is never 0 by hypothesis. It follows that the map $H : S^1 \times I \longrightarrow S^1$ defined by the formula

$$H(z,t) = \frac{k(z,t)}{\|k(z,t)\|}$$

defines a homotopy from z^n to f. This shows that f has degree n.

Combining the two statements gives that $n = 0$, so that p is a constant polynomial. ■

Application: ...everything?