Math 751 - Vector Bundles
Homework 1
Fall 2018

1. Let $p : E \longrightarrow B$ be a vector bundle of rank n. Suppose that $U, V \subseteq B$ are overlapping open subsets on which the bundle is trivial. Then the composition

$$(U \cap V) \times \mathbb{R}^n \xrightarrow{\varphi_U^{-1}} p^{-1}(U \cap V) \xrightarrow{\varphi_V} (U \cap V) \times \mathbb{R}^n$$

is of the form

$$(x, \mathbf{v}) \mapsto (x, g_{U,V}(x)\mathbf{v}),$$

where $g_{U,V} : U \cap V \longrightarrow Gl_n(\mathbb{R})$ is continuous. (These are called **transition functions**.) Show that these transition functions satisfy

   (a) $g_{U,U} = \mathrm{id}$

   (b) $g_{V,U} = g_{U,V}^{-1}$

   (c) $g_{V,W}g_{U,V} = g_{U,W}$.

2. Let B have a covering by open sets $\{U_\alpha\}$ and let $\{g_{\alpha,\beta} : U_\alpha \cap U_\beta \longrightarrow Gl_n(\mathbb{R})\}$ be continuous maps satisfying (a)-(c) above. Define E by

$$E :\, = \big(\coprod_\alpha U_\alpha \times \mathbb{R}^n\big) / \sim,$$

where $(x, \mathbf{v}) \sim (x, g_{\alpha,\beta}(\mathbf{v}))$ for $x \in U_\alpha \cap U_\beta$. Fill in the details to make E into a rank n vector bundle over B.

3. (a) Let $p \in \mathbb{R}^n$. According to Taylor's theorem (in $\mathbb{R}^n$), any smooth f can be written

$$f(\mathbf{x}) = f(\mathbf{p}) + \sum_i \frac{\partial f}{\partial x_i}(\mathbf{p})(x_i - p_i) + R_1^f(\mathbf{x}),$$

   where the remainder R_1^f is of the form

$$R_1^f(\mathbf{x}) = \sum_{i,j}(x_i - p_i)(x_j - p_j)g(\mathbf{x}).$$

   Use this to show that if λ is a derivation at p on $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, then

$$\lambda = \sum_{i=1}^n \lambda_i \frac{\partial}{\partial x_i}\Big|_{x=p},$$

   where $\lambda_i = \lambda(x_i)$ and $x_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the ith coordinate function.

   (b) Conclude that the map $\mathbf{v} \mapsto \partial_{\mathbf{v}}(-)$, defined just above Definition 1.11 in the notes, is an isomorphism from the space of geometric tangent vectors (at p) in $\mathbb{R}^n$ to the space of derivations (at p) on $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$.