Math 751 - Vector Bundles
Worksheet 1
Fall 2018

1. Write down (global) trivializations for the tangent and normal bundle of S^1.
2. Try to repeat problem (1) for S^2 and S^3.
3. Let M be a smooth manifold and $p \in M$. Let $\mathfrak{m} \subseteq \mathcal{C}^\infty(M, \mathbb{R})$ be the (maximal) ideal consisting of functions that vanish at p. Consider the assignment

$$\chi : \mathrm{Der}_p(M) \longrightarrow (\mathfrak{m}/\mathfrak{m}^2)^{\vee} \quad \text{(dual vector space)}$$

given by

$$\chi(\lambda)(f) = \lambda(f).$$

   (a) Show that χ is a well-defined linear map.

   (b) Show that χ is injective. (Hint: You will need to work out how derivations act on constant functions.)

   (c) Show that χ is surjective. (Hint: Given $\phi \in (\mathfrak{m}/\mathfrak{m}^2)^{\vee}$, define λ_ϕ by the formula $\lambda_\phi(f) = \phi(f - f(p))$.)

Problem 4 was moved to homework.

4. (a) Let $p \in \mathbb{R}^n$. According to Taylor's theorem (in $\mathbb{R}^n$), any smooth f can be written

$$f(\mathbf{x}) = f(\mathbf{p}) + \sum_i \frac{\partial f}{\partial x_i}(\mathbf{p})(x_i - p_i) + R_1^f(\mathbf{x}),$$

   where the remainder R_1^f is of the form

$$R_1^f(\mathbf{x}) = \sum_{i,j} (x_i - p_i)(x_j - p_j) g(\mathbf{x}).$$

   Use this to show that if λ is a derivation at p on $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, then

$$\lambda = \sum_{i=1}^{n} \lambda_i \frac{\partial}{\partial x_i}\Big|_{x=p},$$

   where $\lambda_i = \lambda(x_i)$ and $x_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the ith coordinate function.

   (b) Conclude that the map $\mathbf{v} \mapsto \partial_{\mathbf{v}}(-)$, defined just above Definition 1.11 in the notes, is an isomorphism from the space of geometric tangent vectors (at p) in $\mathbb{R}^n$ to the space of derivations (at p) on $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$.