Math 751 - Vector Bundles
Worksheet 10
Fall 2018

1. Let $x = \sum_{i=0}^{n} x_n$ be an element of $\mathrm{H}^*(X; \mathbf{F}_2)$, where $x_0 = 1$ and the degree of x_i is i. Show x is invertible in $\mathrm{H}^*(X; \mathbf{F}_2)$.

2. Let γ_n be the canonical line bundle over $\mathbb{RP}^n$. Note that γ_n is defined as a subbundle of $\underline{n+1}$. Let E be the orthogonal complement of γ_n. Find the total Stiefel-Whitney class of E.

3. Recall that if $M \subseteq N$ is a submanifold, then the tangent bundle τ_N of N, when restricted to the submanifold M, splits as $(\tau_N)_{|M} \cong \tau_M \oplus \nu$, where ν is the normal bundle to the embedding $M \hookrightarrow N$.

   We showed in class that $w(\tau_{\mathbb{RP}^n}) = (1+x)^{n+1}$. Find $w(\nu)$, where ν is the embedding $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1}$. Which bundle is ν?

4. An embedding $M \hookrightarrow N$ (or more generally, immersion) induces an injection on tangent bundles. For example, an immerion $\mathbb{RP}^n \hookrightarrow \mathbb{R}^n + k$ gives an inclusion $\tau_{\mathbb{RP}^n} \hookrightarrow \underline{n+k}$. Use the total Stiefel-Whitney class of $\tau_{\mathbb{RP}^n}$ and the normal bundle ν to establish the following lower bounds on k:

   (a) When $n = 4$, then $k \geq 3$.

   (b) When $n = 8$, then $k \geq 7$.

   (c) More generally, when $n = 2^\ell$, then $k \geq 2^\ell - 1$.