Math 751 - Vector Bundles
Worksheet 11
Fall 2018

1. We established a bijection between the number of (Schubert) cells of $Gr_n(\mathbb{R}^k)$ of dimension r and the number of (unordered) partitions of r into at most n positive integers, each of which is $\leq k-n$,
$$(s_1, \ldots, s_n) \leftrightarrow \{s_1 - 1, \ldots, s_n - n\},$$
where on the right we ignore any zeros.

   Consider the inclusion $Gr_n(\mathbb{R}^k) \hookrightarrow Gr_{n+1}(\mathbb{R}^{k+1})$ given by $X \mapsto \mathbb{R} \oplus X$. What is the image of a Schubert cell $e(s_1, \ldots, s_n)$ under this inclusion? What about under the inclusion $X \mapsto X \oplus \mathbb{R}$?

2. (a) Use the fact $\mathbb{RP}^\infty \vee \mathbb{RP}^\infty \hookrightarrow \mathbb{RP}^\infty \times \mathbb{RP}^\infty$ is an isomorphism on $\mathrm{H}^1(-;\mathbb{F}_2)$ to conclude that
$$w_1(p_1^*\gamma \otimes p_2^*\gamma) = w_1(p_1^*\gamma) + w_1(p_2^*\gamma).$$
Conclude that for any two line bundles L_1 and L_2 on an arbitrary space,
$$w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2).$$

   (b) Suppose that E is a rank 2 bundle and F is rank 1. Find the total Stiefel-Whitney class $w(E \otimes F)$.

   (Hint: by the splitting principle, it suffices to consider the case where E splits $E \cong L_1 \oplus L_2$ as a sum of line bundles.)

   (c) Try the same problem when E is rank 3.

3. A theorem of Borel states that
$$\mathrm{H}^*(Gr_n(\mathbb{R}^k);\mathbb{F}_2) \cong \mathbb{F}_2[w_1, \ldots, w_n]/I_{n,k},$$
where $I_{n,k}$ is the ideal generated by the homogeneous components of $w(\gamma^n)^{-1}$ in degrees $k-n+1, \ldots, k$. Compute
$$\mathrm{H}^*(Gr_2(\mathbb{R}^3);\mathbb{F}_2) \qquad \text{and} \qquad \mathrm{H}^*(Gr_2(\mathbb{R}^4);\mathbb{F}_2).$$