Math 751 - Vector Bundles
Worksheet 5
Fall 2018

1. Recall the group completion construction: if M is an abelian semigroup, we define

$$\widetilde{M} = \mathbb{Z}\{M\}/\sim, \qquad [m_1] + [m_2] \sim [m_1 + m_2].$$

   Show that this satisfies the following universal property: any homomorphism from M to an abelian group A factors uniquely through $\widetilde{M}$. In other words, show that

$$\widetilde{(-)} : \mathbf{AbSemGp} \longrightarrow \mathbf{AbGp}$$

   is *left adjoint* to the inclusion $\mathbf{AbGp} \hookrightarrow \mathbf{AbSemGp}$.

2. Let M be an abelian semigroup. Let $\widehat{M} := (M \times M)/\sim$, where

$$(m_1, m_2) \sim (n_1, n_2) \qquad if \qquad m_1 + n_2 + k = n_1 + m_2 + k,$$

   for some $k \in M$. The ordered pair (m_1, m_2) plays the role of $m_1 - m_2$. Show that $\widehat{M} \cong \widetilde{M}$.

3. Show that if A is a semiring then $\widetilde{A}$ is a ring.

4. Compute $KO(*)$ and $KU(*)$.

5. Consider the set $\mathbf{Set}_{C_2}$ of (isomorphism classes) of finite C_2-sets. This is a semiring under disjoint union and cartesion product. Write $A(C_2)$ for the group completion, which is known as the **Burnside ring** of C_2. Describe this ring explicitly (as a quotient of a polynomial ring, for example).