Math 751 - Vector Bundles
Worksheet 6
Fall 2018

1. Suppose that $F\colon \mathscr{C} \rightleftarrows \mathscr{D}\colon G$ is an adjoint pair of functors. Use this to construct natural transformations
$$\mathrm{id}_{\mathscr{C}} \xrightarrow{\eta} GF, \qquad FG \xrightarrow{\varepsilon} \mathrm{id}_{\mathscr{D}}$$
satisfying the "triangle identities"
$$G\varepsilon \circ \eta G = \mathrm{id}_G \qquad \text{and} \qquad \varepsilon F \circ F\eta = \mathrm{id}_F,$$
which can be represented by the commutative diagrams
$$\begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle G\varepsilon} \\ & & G \end{array} \qquad \text{and} \qquad \begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \varepsilon F} \\ & & G. \end{array}$$

2. Conversely, suppose given functors $F\colon \mathscr{C} \rightleftarrows \mathscr{D}\colon G$ and natural transformations
$$\mathrm{id}_{\mathscr{C}} \xrightarrow{\eta} GF, \qquad FG \xrightarrow{\varepsilon} \mathrm{id}_{\mathscr{D}}$$
satisfying the "triangle identities"
$$G\varepsilon \circ \eta G = \mathrm{id}_G \qquad \text{and} \qquad \varepsilon F \circ F\eta = \mathrm{id}_F\,.$$
Show that this data makes F left adjoint to G.

3. At the end of class on Wednesday, we discussed how the bijection
$$KO(X) \cong [X, BO \times \mathbb{Z}]$$
implies that the space $BO \times \mathbb{Z}$ is a topological ring up to homotopy, meaning that the ring axioms only hold up to homotopy. We discussed what the addition would look like. What are the additive and multiplicative units? What is the additive inverse? What is the multiplication?