Math 751 - Vector Bundles
Worksheet 8
Fall 2018

1. Suppose that S^n is parallelizable (has trivial tangent bundle), with trivialization $\varphi : S^n \times \mathbb{R}^n \cong TS^n$. Given $\mathbf{x} \in S^n$, write

$$p_{\mathbf{x}} : S^n \xrightarrow{\cong} \widehat{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}, \qquad q_{\mathbf{x}} = p_{\mathbf{x}}^{-1} : \widehat{\mathbb{R}^n} \xrightarrow{\cong} S^n$$

for stereographic projection and its inverse, so that $p_{\mathbf{x}}(-\mathbf{x}) = \mathbf{0}$ and $p_{\mathbf{x}}(\mathbf{x}) = \infty$.

Define $\varphi : TS^n \longrightarrow S^n$ by letting $\varphi(\mathbf{x}, \mathbf{y})$ be the point on the line through $-\mathbf{x}$ and $\mathbf{x} + \mathbf{y}$ (recall that $\mathbf{y} \perp \mathbf{x}$) that lies on S^n. Define $\nu : S^n \times \mathbb{R}^n \longrightarrow S^n$ by the composition

$$S^n \times \mathbb{R}^n \cong TS^n \xrightarrow{\varphi} S^n.$$

Then this extends to a map $\overline{\nu} : S^n \times \widehat{\mathbb{R}^n} \longrightarrow S^n$. Now pick some $\mathbf{e} \in S^n$ and define $\mu : S^n \times S^n \longrightarrow S^n$ by the formula

$$S^n \times S^n \xrightarrow{\mathrm{id} \times p_{-\mathbf{e}}} S^n \times \widehat{\mathbb{R}^n} \xrightarrow{\overline{\nu}} S^n.$$

Show that this makes S^n an H-space with unit $\mathbf{e}$. (The Hopf invariant one theorem now implies that S^n can only be parallelizable when $n = 0, 1, 3, 7$.)

2. Suppose that $\mathbb{R}^n$ is a division algebra (has a bilinear product μ with no zero divisors). Pick a nonzero vector $\mathbf{e} \in \mathbb{R}^n$. Show that there is a division algebra structure in which $\mathbf{e}$ is the unit. Pick a linear isomorphism $f : \mathbb{R}^n \cong \mathbb{R}^n$ such that $f(\mu(\mathbf{e}, \mathbf{e})) = \mathbf{e}$. Then $\overline{\mu} = f \circ \mu$ takes $(\mathbf{e}, \mathbf{e})$ to $\mathbf{e}$. Write $\ell(\mathbf{x}) = \overline{\mu}(\mathbf{x}, \mathbf{e})$ and $r(\mathbf{x}) = \overline{\mu}(\mathbf{e}, \mathbf{x})$. Show that

$$\nu(\mathbf{x}, \mathbf{y}) = \overline{\mu}(\ell^{-1}(\mathbf{x}), r^{-1}(\mathbf{y}))$$

is a division algebra structure on $\mathbb{R}^n$ with unit element $\mathbf{e}$.

3. Consider the function $c : \mathrm{Vect}_{\mathbb{R}}(X) \longrightarrow \mathrm{Vect}_{\mathbb{C}}(X)$ which takes a real bundle and tensors each fiber with $\mathbb{C}$ to produce a complex bundle. This map induces a map on K-theory which fits into a long exact sequence (I am not asking you to prove this part) of the form

$$\cdots \xrightarrow{c} \widetilde{KU}^{n-1}(X) \longrightarrow \widetilde{KO}^{n+1}(X) \longrightarrow \widetilde{KO}^{n}(X) \xrightarrow{c} \widetilde{KU}^{n}(X) \longrightarrow \cdots.$$

Use Bott periodicity to analyze this long exact sequence in the case $X = S^0$.