Math 654 - Algebraic Topology
Homework 4
Fall 2019

1. Recall the functor $\mathrm{Gl}_n : \mathbf{Comm} \to \mathbf{Gp}$ from HW1. When $n = 1$, this gives the functor $(-)^\times : \mathbf{Comm} \to \mathbf{Gp}$ which takes a commutative ring R and gives $R^\times$, the units (invertible elements) in R. Show that the determinant yields a natural transformation $\det : \mathrm{Gl}_n \to (-)^\times$.

2. Let G be a group. Define a category $\star_G$ which has a single object, $\star$, and such that $\mathrm{Hom}(\star, \star) = G$. The identity morphism and composition of morphisms are defined to be the identity element of the group and the group multiplication, respectively.

   (a) Show that a G-set X is the same data as a functor $\mathcal{X} : \star_G \longrightarrow \mathbf{Set}$.

   (b) Show that if X and Y are G-sets, then a G-equivariant function $f : X \longrightarrow Y$ corresponds precisely to a natural transformation of functors $\mathcal{X} \longrightarrow \mathcal{Y}$.

3. Let $\mathcal{I} = \{\bullet \longrightarrow \bullet\}$ be the category with two objects and a single non-identity morphism. Describe the data involved in a natural transformation $\eta : F \Rightarrow G : \mathcal{I} \longrightarrow \mathscr{C}$.

4. Let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a functor. Let $G : \mathrm{Ob}(\mathscr{C}) \longrightarrow \mathrm{Ob}(\mathscr{D})$ be a function, and suppose given an isomorphism $\eta_C : F(C) \cong G(C)$ for each $C \in \mathscr{C}$. Show that there is a unique way to define G on morphisms of $\mathscr{C}$ that makes $\{\eta_C\}$ a natural isomorphism.