Math 654 - Algebraic Topology
Homework 5
Fall 2019

1. A chain map $f_* : C_* \longrightarrow D_*$ that induces an isomorphism in homology is called a **quasi-isomorphism**. We showed in class that any chain homotopy equivalence is a quasi-isomorphism. Give an example of a quasi-isomorphism of chain complexes which is not a chain homotopy equivalance.

   (Hint: Find a quasi-isomorphism $f_* \colon C_* \longrightarrow D_*$ for which there are no nonzero chain maps $D_* \longrightarrow C_*$.)

2. Show that the chain complex $C_*^{\Delta}(\mathbb{RP}^2)$ described in class (on 9-9-19) is chain homotopy equivalent to the complex $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$.

3. Recall that a **short exact sequence** of abelian groups is a sequence of homomorphisms

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

   which is exact (has trivial homology) at each spot. A short exact sequence is called **split exact** if $B \cong A \oplus C$. Show that the following are equivalent for the (solid arrow) sequence:

$$0 \longrightarrow A \underset{r}{\overset{i}{\rightleftarrows}} B \underset{s}{\overset{p}{\rightleftarrows}} C \longrightarrow 0$$

   (a) The sequence is split exact

   (b) There exists a homomorphism s such that $p \circ s = \mathrm{id}_C$ (s is called a splitting)

   (c) There exists a homomorphism r such that $r \circ i = \mathrm{id}_A$ (r is called a retraction or splitting)