CLASS NOTES
EQUIVARIANT HOMOTOPY AND COHOMOLOGY
MATH 751 (FALL 2020)

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CONTENTS

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1. Equivariance in Algebra

Mon, Aug. 17

Throughout this course, G will denote a finite group. Some important examples will be

(1) **cyclic groups**: $(C_2, C_3, \dots)$,
(2) **elementary abelian groups**: meaning products $(C_p)^{\times n}$, such as the Klein four group $K_4 = C_2 \times C_2$,
(3) **dihedral groups**: we will denote by D_n the dihedral group of order $2n$,
(4) **symmetric groups**: we will follow the topological convention of denoting these by Σ_n.

Thus we have inclusions

$$1 \leq C_n \leq D_n \leq \Sigma_n$$

(the latter assuming that $n \geq 3$).

1.1. **Group actions in algebra: representations.**

One of the central notions in this course is that of a group action. Let's first quickly review what this means in the most general possible setting. This means we want the categorical definition. Recall that if X is an object of some category $\mathcal{C}$, then the set of all morphisms $X \longrightarrow X$ in $\mathcal{C}$ forms a monoid (meaning a group, possibly without inverses) under composition, called $\mathrm{End}(X)$.

Definition 1.1.1. Let X be an object of some category $\mathcal{C}$. An action of a group G on X consists of a homomorphism of monoids $G \xrightarrow{a} \mathrm{End}(X)$.

Equivalently, we can describe a group action on X as a group homomorphism $G \xrightarrow{a} \mathrm{Aut}(X)$ from G to the group of automorphisms of X.

Unpacking the definition, this means that

(1) for each $g \in G$, we have an morphism $a(g) \colon X \longrightarrow X$,
(2) the function a preserves composition, so that $a(g \cdot h) = a(g) \circ a(h)$, and
(3) the function a preserves identities, so that $a(e) = \mathrm{id}_X$.

In many situations, there is an "adjoint" formulation.

Watch the video (in canvas): adjoint functors!

Consider one of the simplest situations, where $\mathcal{C}$ is the category **Set** of sets and functions. Then

$$a \colon G \longrightarrow \mathrm{End}(X) = \{f \colon X \longrightarrow X\}$$

corresponds to a function

$$G \times X \longrightarrow X.$$

Then conditions (1) and (2) above turn into the conditions that the diagrams

$$\begin{tikzcd} G \times G \times X \arrow[r, "m \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times a"'] & G \times X \arrow[d, "a"] \\ G \times X \arrow[r, "a"] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} * \times X \arrow[r, "e \times \mathrm{id}"] \arrow[d, "\cong"'] & G \times X \arrow[d, "a"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

both commute. Typically, we write $g \cdot x$ for the action of g on x.

Implementing these ideas in algebra, we take $\mathcal{C}$ to be the category $\mathbf{Mod}_R$ of modules over a commutative ring R. In practice, R will be either the universal case $R = \mathbb{Z}$, or it will be a field $\mathbf{k}$. For $M \in \mathbf{Mod}_R$, an action of G on M would mean a homomorphism

$$a\colon G \longrightarrow \mathrm{Hom}_R(M, M),$$

where we are considering $\mathrm{Hom}_R(M, M)$ as a monoid under composition of homomorphisms. Notice that the target $\mathrm{Hom}_R(M, M)$ is not only a monoid under composition, it is also an abelian group under addition. These two structures combine to give $\mathrm{Hom}_R(M, M)$ the structure of a (noncommutative) ring.

Even better, the (additive) abelian group extends canonically to an R-module, and this makes $\mathrm{Hom}_R(M, M)$ an R-algebra. We can then get an R-linear version of the homomorphism a as follows. Writing $R[G]$ for a free R-module with generators given by the elements of G, the group structure on G endows the R-module $R[G]$ with the structure of an R-algebra. This is the **R-linear group ring** on G.

The above discussion leads to

Proposition 1.1.2. *Let R be a commutative ring and G a group. The following are equivalent data on an R-module M:*

(1) *a monoid homomorphism $G \longrightarrow \mathrm{End}_R(M)$,*
(2) *a group homomorphism $G \longrightarrow \mathrm{Aut}_R(M)$,*
(3) *a homomorphism of R-algebras $R[G] \longrightarrow \mathrm{Hom}_R(M, M)$*
(4) *an $R[G]$-module structure on M whose underlying R-module is M.*

Proof. Exercise. ■

Definition 1.1.3. A **representation of** G over a commutative ring R is an $R[G]$-module.

Typically, this language is used in the case that $R = \mathbf{k}$ is a field. Then the underlying R-module is simply a $\mathbf{k}$-vector space V. And if $\dim_{\mathbf{k}} V = n$, then $\mathrm{Aut}_{\mathbf{k}}(V)$ is usually written $\mathrm{Gl}_n(\mathbf{k})$. In other words, an n-dimensional representation of G over $\mathbf{k}$ can equivalently be described as a group homomorphism $G \longrightarrow \mathrm{Gl}_n(\mathbf{k})$.

Example 1.1.4. The $R[G]$-module $R[G]$ itself is known as the **regular representation**.

Example 1.1.5. More generally, given any finite G-set X, the free R-module $R[X]$ inherits the structure of an $R[G]$-module. In the language of representation theory, representations arising in this way are known as **permutation representations**.

Example 1.1.6. An important example of a permutation representation is a **trivial representation**. The one-dimensional trivial representation is R, where $g \cdot r = r$ for all $g \in G$ and $r \in R$. In general, a trivial representation is a direct sum of copies of the one-dimensional trivial representation.

Example 1.1.7. Let $G = C_2 = \langle \tau \rangle$, and suppose that -1 is not equal to 1 in R (in other words, think $R = \mathbb{Z}$ or a field of characteristic not equal to 2). The **sign representation** of $G = C_2$ is the one-dimensional representation in which the element τ acts as -1.

Notice that this definition still makes sense if R is of characteristic 2, but then the sign representation just becomes the trivial representation. Note also that the sign representation is our first example of a representation that is **not** a permutation representation.

Example 1.1.8. Let $G = C_n$, for $n \geq 3$. We define the two-dimensional real representation λ_n by letting the generator of C_n act as rotation by an angle of $\frac{2\pi}{n}$. Note that we can equally well treat this as a one-dimensional complex representation, in which the generator of C_n acts as multiplication by the nth root of unity $\zeta_n = e^{\frac{2\pi}{n} i}$.

Example 1.1.9. For any R, we can define an n-dimensional representation of the symmetric group Σ_n, by letting the symmetric group act on R^n by permuting the coordinates. In the case that R is $\mathbb{R}$ or $\mathbb{C}$, this is known as the **standard representation** of Σ_n.

$\boxed{\textbf{Wed, Aug. 19}}$

Since the regular and trivial representations show up so frequently, we introduce convenient notation for them:

Notation 1.1.10. We will denote the regular representation for G by $\rho_G = R[G]$ and the trivial representation by $\mathbf{n} = R^{\oplus n}$.

In both cases, the ground ring R is not denoted, but this will usually be clear from context.

Since a representation is the same as a module over the group ring $R[G]$, we can define a subrepresentation to simply be a submodule.

Example 1.1.11. The regular representation $R[G]$ always has a one-dimensional trivial subrepresentation, generated by the sum $\sum_{g \in G} g$.

Example 1.1.12. Take R to be $\mathbb{Z}$ or a field of characteristic not equal to 2. Then the submodule $R\{1 - \tau\} \subset R[C_2]$ is isomorphic to the sign representation.

1.1.1. *Irreducible representations and Maschke's Theorem.* In module theory, a simple module is a module with no nonzero, proper submodules. These play an important role in representation theory.

Definition 1.1.13. We say that V is an **irreducible** representation if the only subrepresentations of V are 0 and V.

Clearly, any one-dimensional representation over a field must be irreducible, since a one-dimensional vector space has no nonzero proper sub-vector space. When $R = \mathbb{C}$, then in fact all irreducibles are one-dimensional, but over other fields, this is not the case.

Example 1.1.14. Consider the (real) two-dimensional rotation representation λ_3 of $C_3 = \{e, r, r^2\}$ (Example 1.1.8). Since rotation by $\frac{2\pi}{3}$ does not send any nonzero vector to a (real) scalar multiple of itself, λ_3 does not have a nonzero subrepresentation. Said differently, the reason is that the matrix of rotation by $\frac{2\pi}{3}$ does not have any real eigenvaules.

The following result is very useful. Mostly, we will use this for the fields $\mathbf{k} = \mathbb{R}$ and $\mathbb{C}$ of characteristic zero.

Theorem 1.1.15 (Maschke)**.** *Suppose that $\mathbf{k}$ is a field of characteristic not dividing $|G|$. Then every representation splits as a direct sum of irreducible representations.*

Proof. It suffices to show that if $V \subset W$ is a subrepresentation, then there exists a subrepresentation $U \subset W$ and an isomorphism $U \oplus V \cong W$. Let $p \colon W \longrightarrow W$ be any linear projection onto the subspace V. There is no reason for p to be G-equivariant, but we can modify it to make it so. This is the key idea in the proof. Define a new linear map $\varphi \colon W \longrightarrow W$ by the formula

$$\varphi(\mathbf{w}) = \frac{1}{|G|} \sum_{g \in G} g \cdot p(g^{-1} \cdot \mathbf{w}).$$

Then φ is G-equivariant, in the sense that

$$\varphi(h \cdot \mathbf{w}) = h \cdot \varphi(\mathbf{w})$$

for all $\mathbf{w} \in W$ and $h \in G$. Since p fixes the subspace V (meaning that $p(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v}$ in V) and V is G-invariant (meaning that $g \cdot \mathbf{v}$ lies in V for all $\mathbf{v} \in V$) it follows that φ also fixes V. Since the image of p is V, it follows that same is true of φ. Combining these two statements, we learn that φ is idempotent, meaning that $\varphi \circ \varphi = \varphi$.

Now we just use general properties of idempotents: since φ is an idempotent with image equal to V, if we set $U = \ker(\varphi)$, then it follows that $W \cong V \oplus U$. ■

The restriction on the characteristic was used in the proof to know that $|G| \neq 0$ in $\mathbf{k}$. This requirement is necessary, as the next example shows.

Example 1.1.16. Consider the regular representation for $G = C_2$, but in the case $R = \mathbb{F}_2$. As we mentioned in Example 1.1.11 above, the regular representation contains a one-dimensional trivial subrepresentation. But we claim that in this case the trivial subrepresentation does not split off. Consider a C_2-equivariant map $p\colon \mathbb{F}_2[C_2] \longrightarrow \mathbf{1}$. Such a map is completely determined by the vaue $p(1)$.

There are two choices: either $p(1) = 0$, in which case p is the zero map, or $p(1) = 1$, in which case equivariance forces p to be given by the formula $p(a + b\tau) = a + b$. The kernel of this homomorphism is precisely the trivial subrepresentation. This shows that there is no projection from the regular representation to its trivial subrepresentation.

Despite this example, we will mostly focus on the case of real representations, so that Maschke's theorem will allow us to decompose any representation as a sum of irreducibles.

One important consequence of Maschke's theorem is that all irreducibles show up as submodules of the regular representation:

Proposition 1.1.17. *Suppose that* $\mathbf{k}$ *is a field of characteristic not dividing* $|G|$*. If* V *is an irreducible representation, then* V *is isomorphic to a submodule of* $\mathbf{k}[G]$*.*

Proof. Let $\mathbf{v} \in V$ be nonzero. Then the homomorpihsm $\varphi\colon \mathbf{k}[G] \longrightarrow V$ sending 1 to $\mathbf{v}$ must be surjective (else its image would be a nonzero submodule, contradicting that V is irreducible). Letting $U = \ker(\varphi)$, Maschke's theorem tells us that U splits off, and the isomorphism theorems tell us that the complement is isomorphic to V. ∎

Example 1.1.18. In the case $G = C_2$, we have the trivial representation $\mathbf{1}$ and the sign representation $\mathbf{1}_{\text{sgn}}$. As we have already said, the trivial subrepresentation is generated by $1 + \tau$, while the sign representation is generated by $1 - \tau$. Thus

$$\rho_{C_2} \cong \mathbf{1} \oplus \mathbf{1}_{\text{sgn}}.$$

Example 1.1.19. In the case $G = C_3$, we again have the trivial representation $\mathbf{1} \subset \rho_{C_3}$, and the complement is necessarily the irreducible two-dimensional representation λ_3. We have

$$\rho_{C_3} \cong \mathbf{1} \oplus \lambda_3.$$

Note that if we denote by $\overline{\lambda_3}$ the representation in which the generator acts by a clockwise rotation instead, then $\overline{\lambda_3}$ is also two-dimensional, but in fact reflection across the x-axis produces an isomorphism $\lambda_3 \cong \overline{\lambda_3}$.

Example 1.1.20. Consider now $G = C_5$. In addition to the trivial representation $\mathbf{1}$ and the two-dimensional representation λ_5, we also have the two-dimensional representation in which the generator acts by rotation by an angle $2 \cdot \frac{2\pi}{5}$. We will write $\lambda_{5,2}$ for this representation. Then we have

$$\rho_{C_5} \cong \mathbf{1} \oplus \lambda_5 \oplus \lambda_{5,2}.$$

Watch the video (in canvas): Representations of the dihedral group.

Mon, Aug. 24

1.1.2. *The representation ring.* Last week, we discussed representations. And just as we can take direct sums and tensor products of vector spaces, the same constructions exist at the level of representations.

The direct sum is simpler. If V and W are representations for G, then we define $V \oplus W$ to be the underlying direct sum of vector spaces, and where the G-action is given by

$$g \cdot (v, w) = (g \cdot v, g \cdot w).$$

Viewing a representation as a homomorphism $G \longrightarrow \mathrm{Gl}_n(\mathbf{k})$, this amounts to taking the block sum of matrics:

$$G \longrightarrow \mathrm{Gl}_n(\mathbf{k}) \times \mathrm{Gl}_k(\mathbf{k}) \xrightarrow{\oplus} \mathrm{Gl}_{n+k}(\mathbf{k}).$$

We can take the same perspective to define the tensor product:

$$G \longrightarrow \mathrm{Gl}_n(\mathbf{k}) \times \mathrm{Gl}_k(\mathbf{k}) \xrightarrow{\otimes} \mathrm{Gl}_{nk}(\mathbf{k}).$$

Thinking in terms of elements, the action on a simple tensor looks like

$$g \cdot (\mathbf{v} \otimes \mathbf{w}) = g \cdot \mathbf{v} \otimes g \cdot \mathbf{w}.$$

Here the underlying vector space of a tensor product of representations $V \otimes W$ is just the tensor product of vector spaces $V \otimes_{\mathbf{k}} W$.

Example 1.1.21. The trivial representation $\mathbf{1}$ acts like a multiplicative unit, in the sense that

$$\mathbf{1} \otimes V \cong V \cong V \otimes \mathbf{1}$$

for any V. Combining this with the distributive law shows that

$$\mathbf{n} \otimes V \cong V^{\oplus n} \cong V \otimes \mathbf{n}$$

for any V.

Example 1.1.22. Consider the tensor product $\mathbf{1}_{\mathrm{sgn}} \otimes \mathbf{1}_{\mathrm{sgn}}$. The underlying vector space is one-dimensional, and the element τ acts on the basis element as

$$\tau \cdot (1 \otimes 1) = \tau \cdot 1 \otimes \tau \cdot 1 = -1 \otimes -1 = 1 \otimes 1,$$

which shows that $\mathbf{1}_{\mathrm{sgn}} \otimes \mathbf{1}_{\mathrm{sgn}}$ is the trivial representation $\mathbf{1}$.

In general, it is not easy to understand a tensor product $V \otimes W$ of representations.

Example 1.1.23. Take $G = C_3$ and recall the two-dimensional rotation representation λ_3 (Example 1.1.8). We know that $\lambda_3 \otimes \lambda_3$ is a four-dimensional representation, and so there are only three possibilities for its decomposition into irreducibles: either $\mathbf{4}$ or $\mathbf{2} \oplus \lambda_3$ or $\lambda_3^{\oplus 2}$. In fact, the answer is $\mathbf{2} \oplus \lambda_3$.

Watch the video (in canvas): The tensor product $\lambda_3 \otimes \lambda_3$

We have discussed two operations on representations: direct sum and tensor product. If we pass to isomorphism classes, this makes the set of (finite-dimensional) representations into a commutative sermiring. It is only a semiring because there are no additive inverses. There is a process for "group completing" a semiring into a ring, known variously as the Grothendieck construction, or K_0-group, or simply group completion.

Definition 1.1.24. For a finite group G, the **real representation ring** of G, denoted $RO(G)$, is the quotient

$$RO(G) = \mathbb{Z}\left\{\begin{matrix}\text{isomorphism classes of finite-}\\ \text{dimensional real } G\text{-representations}\end{matrix}\right\} / \langle [V \oplus W] - [V] - [W] \rangle.$$

In this case, since real representations decompose (canonically) into a direct sum of irreducibles, we can get away with this simpler definition: writing $\mathrm{Irrep}(G)$ for the set of isomorphism classes of irreducible G-representations, we have

$$RO(G) \cong \bigoplus_{\mathrm{Irrep(G)}} \mathbb{Z}.$$

Example 1.1.25. For $G = C_2$, we have

$$RO(C_2) = \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\mathbf{1}_{\mathrm{sgn}}\},$$

where $(\mathbf{1}_{\mathrm{sgn}})^2 = \mathbf{1}$. In other words, we can describe this ring as

$$RO(C_2) \cong \mathbb{Z}[\sigma]/(\sigma^2 - 1).$$

Example 1.1.26. For $G = C_3$, we have

$$RO(C_3) = \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\lambda_3\},$$

where $[\lambda_3]^2 = 2 + [\lambda_3]$ according to Example 1.1.23. In other words, we can describe this ring as

$$RO(C_3) \cong \mathbb{Z}[\lambda]/(\lambda^2 - \lambda - 2).$$

Notice that in both of these examples, the representation ring has zero divisors. For example, in $RO(C_2)$, we have that $(\sigma - 1)(\sigma + 1) = 0$, but neither $\sigma - 1$ nor $\sigma + 1$ is equal to zero.

1.1.3. *Pullback of representations.* We now turn our attention to change-of-group constructions. The first such construction uses the idea that, since a G-representation corresponds to a homomorphism $G \longrightarrow \mathrm{Aut}(V)$ for some vector space V, we can compose this with any homomorphism $H \longrightarrow G$ to obtain an H-representation.

Definition 1.1.27. Let $\varphi\colon H \twoheadrightarrow G$ be a group homomorphism. We define the **pullback** $\varphi^*(V)$ of a G-representation along φ to be the same underlying vector space and with action given by $H \xrightarrow{\varphi} G \xrightarrow{a} \mathrm{End}(V)$.

Pullback interacts well with respect to direct sum and tensor product, meaning that

$$\varphi^*(V \oplus W) \cong \varphi^*(V) \oplus \varphi^*(W) \qquad \text{and} \qquad \varphi^*(V \otimes W) \cong \varphi^*(V) \otimes \varphi^*(W).$$

As a result, we get

Corollary 1.1.28. *For any group homomorphism $\varphi\colon H \longrightarrow G$, the pullback defines a ring homomorphism*

$$\varphi^*\colon RO(G) \longrightarrow RO(H).$$

Now any homomorphism $\varphi\colon H \longrightarrow G$ has a canonical factorization

$$H \twoheadrightarrow K = \mathrm{im}(\varphi) \hookrightarrow G,$$

and so it suffices to consider the two cases of injective and surjective homomorphisms. We first consider the surjective case (quotients).

Example 1.1.29. In the extreme case of $G \twoheadrightarrow G/G = \langle e \rangle$, the pullback of an n-dimensional vector space is the trivial representation $\mathbf{n} = q^*(\mathbf{k}^n)$.

Example 1.1.30. Recall that any dihedral group has a cyclic subgroup of index two. Thus we have a quotient map $D_n \twoheadrightarrow C_2$, so that we may pull back the sign representation to define a sign representation for any dihedral group.

Example 1.1.31. In the Dihedral Representations video, we discussed three sign representations for the Klein four group $K_4 = C_2 \times C_2$. These are pulled back along the three quotient maps $K_4 \twoheadrightarrow C_2$.

Wed, Aug. 26

1.1.4. *Restriction of representations.* In Definition 1.1.27, we introduced the idea of the pullback of a representation along a homomorphism. We looked at a few examples in which the homomorphism is a quotient. The other case of interest is that of an inclusion of a subgroup. Thus if $H \le G$ and V is a G-representation, we can "restrict" the action to H to consider V as an H-representation.

Definition 1.1.32. Let V be a G-representation, and let $\iota\colon H \hookrightarrow G$ be the inclusion of a subgroup. We define the **restriction of** V **to** H, denoted either $\downarrow_H^G(V)$ or $\mathrm{Res}_H^G V$, as the pullback $\iota^*(V)$.

The results that we mentioned last time also apply in the case of restriction. Namely, restriction to a subgroup defines a ring homomorphism $\downarrow_H^G\colon RO(G) \longrightarrow RO(H)$, and restriction preserves direct sum and tensor product.

Example 1.1.33. In the extreme case that H is the trivial subgroup, the restriction $\downarrow_e^G(V)$ is simply the underlying vector space of V.

Example 1.1.34. Let $G = C_4 = \langle r \rangle$, and let $H = \langle r^2 \rangle \le C_4$ be the order two subgroup. The group C_4 has three irreducible representations: $\mathbf{1}$, the sign representation $\sigma = \mathbf{1}_{\mathrm{sgn}}$, and the rotation λ_4. Notice that the sign representation was defined as the pullback along the quotient $C_4 \twoheadrightarrow C_4/H \cong C_2$, so that restricting σ to H gives the trivial representation. For the representation λ_4, since the generator r of C_4 acts as the rotation by angle $\frac{\pi}{2}$, it follows that r^2 acts as multiplication by -1. In other words, λ_4 restricts to a two-dimensional sign representation $2\sigma = \sigma \oplus \sigma$. Summarizing, we have

$$\begin{aligned} RO(C_4) = \mathbb{Z}\{\mathbf{1}, \sigma, \lambda_4\} &\longrightarrow RO(C_2) = \mathbb{Z}\{\mathbf{1}, \sigma\} \\ \mathbf{1} &\mapsto \mathbf{1}, \\ \sigma &\mapsto \mathbf{1}, \\ \lambda_4 &\mapsto 2\sigma. \end{aligned}$$

We can use this to deduce partial information about the multiplicative structure of $RO(C_4)$. We know that

$$(\lambda_4)^2 \mapsto (2\sigma)^2 = \mathbf{4},$$

so it follows that λ_4^2 must be either $\mathbf{4}$, or $\mathbf{3} \oplus \sigma$, or $\mathbf{2} \oplus 2\sigma$, or $\mathbf{1} \oplus 3\sigma$, or 4σ. We will see in Example 1.1.43 below that in fact $(\lambda_4)^2 = \mathbf{2} \oplus 2\sigma$.

1.1.5. *Induction of representations.* While restriction takes a representation defined over a group G and produces a representation over a subgroup H, there is a construction in the other direction, which takes as input an H-representation and produces a G-representation.

Definition 1.1.35. Let $H \le G$, and let V be an H-representation. We define the **induced representation** of V up to G, denoted either $\uparrow_H^G(V)$ or $\mathrm{Ind}_H^G V$, as the tensor product $\mathbf{k}[G] \otimes_{\mathbf{k}[H]} V$.

As a vector space, the induced representation is $\bigoplus_{G/H} V$. Like restriction, induction interacts well with direct sums:

$$\uparrow_H^G(V_1 \oplus V_2) \cong \uparrow_H^G(V_1) \oplus \uparrow_H^G(V_2).$$

On the other hand, a comparison of dimensions shows that induction cannot possibly preserve tensor products of representations. In other word,

$$\uparrow_H^G\colon RO(H) \longrightarrow RO(G)$$

is a map of abelian groups but not a ring homomorphism.

General associativity properties of the tensor product show the following:

Proposition 1.1.36. *If $H \leq K \leq G$, then for any H-representation V, we have*
$$\uparrow_H^G(V) \cong \uparrow_K^G\left(\uparrow_H^K(V)\right).$$

Example 1.1.37. For any group G, the regular representation is induced up from the trivial subgroup:
$$\rho_G = \mathbb{R}[G] \cong \mathbb{R}[G] \otimes_{\mathbb{R}} \mathbb{R} = \uparrow_e^G(\mathbf{1}).$$
More generally, if $H \leq G$ is a subgroup, then Proposition 1.1.36 shows that $\uparrow_H^G(\rho_H) \cong \rho_G$.

Example 1.1.38. Consider $C_2 \leq C_4$. Then
$$\uparrow_{C_2}^{C_4}(\mathbf{1}) = \mathbb{R}[C_4] \otimes_{\mathbb{R}[C_2]} \mathbf{1} \cong \mathbb{R}[C_4/C_2].$$
In other words, we can think of this as the pullback of the regular representation for the quotient group $C_4/C_2 \cong C_2$ to C_4. We conclude that
$$\uparrow_{C_2}^{C_4}(\mathbf{1}) \cong \mathbf{1} \oplus \sigma. \tag{1.1.39}$$
If we instead induce up the sign representation, we get
$$\uparrow_{C_2}^{C_4}(\sigma) = \mathbb{R}[C_4] \otimes_{\mathbb{R}[C_2]} \sigma.$$
This is the two-dimensional representation λ_4. This can be seen directly, but another way to see this is to use the fact that induction preserves direct sums, together with our knowledge of the induction of regular representations.
$$\uparrow_{C_2}^{C_4}(\mathbf{1}) \oplus \uparrow_{C_2}^{C_4}(\sigma) \cong \uparrow_{C_2}^{C_4}(\mathbf{1} \oplus \sigma) = \uparrow_{C_2}^{C_4}(\rho_{C_2}) = \rho_{C_4} \cong \mathbf{1} \oplus \sigma \oplus \lambda_4.$$
The isomorphism (1.1.39) then implies that $\uparrow_{C_2}^{C_4}(\sigma)$ must be λ_4.

Watch the video (in canvas): Restriction and induction for D_8

Induction was defined as a tensor product. In general, given a ring homomorphism $\varphi\colon R \longrightarrow S$, the tensor product $S \otimes_R -\colon \mathbf{Mod}_R \longrightarrow \mathbf{Mod}_S$ is left adjoint to the pullback $\varphi^*\colon \mathbf{Mod}_S \longrightarrow \mathbf{Mod}_R$. In the parlance of representations, this manifests as the following result.

Proposition 1.1.40. *Let $H \leq G$. Then induction is left adjoint to restriction:*
$$\uparrow_H^G\colon \mathbf{Mod}_{R[H]} \rightleftarrows \mathbf{Mod}_{R[G]} : \downarrow_H^G.$$

Example 1.1.41. For any group G and G-representation V, then maps of G-representations $f\colon \rho_G \longrightarrow V$ correspond to maps of vector spaces $\mathbf{k} \longrightarrow \downarrow_e^G(V)$ to the underlying vector space of V. Since such a map is determined by where it sends $1 \in \mathbf{k}$, we conclude that a map of G-representations $f\colon \rho_G \longrightarrow V$ is determined by its value at 1.

Mon, Aug. 31

Remember: induction points up, restriction points down.

1.1.6. *The projection formula.* We saw that restriction of representations preserves tensor product, but this cannot be true for induction. In fact, there is a formula that describes the induction of certain tensor products. Note that, since the restriction $\downarrow_H^G(-)\colon RO(G) \longrightarrow RO(H)$ is a ring homomorphism, we can view $RO(H)$ as an $RO(G)$-module. It turns out that induction preserves this module structure:

Proposition 1.1.42. *Let $H \le G$. Then induction $\uparrow_H^G(-)\colon RO(H) \longrightarrow RO(G)$ is an $RO(G)$-module homomorphism. This means that for $V \in RO(G)$ and $W \in RO(H)$, we have an isomorphism*

(Projection formula)
$$\uparrow_H^G\Big(\downarrow_H^G(V)\otimes W\Big) \cong V \otimes \uparrow_H^G(W).$$

Proof. We can use that induction is left adjoint to restriction to produce a map of G-representations. A G-equivariant map

$$\uparrow_H^G\Big(\downarrow_H^G(V)\otimes W\Big) \longrightarrow V \otimes \uparrow_H^G(W)$$

corresponds to an H-equivariant map

$$\downarrow_H^G(V)\otimes W \longrightarrow \downarrow_H^G\Big(V \otimes \uparrow_H^G(W)\Big) \cong \downarrow_H^G(V) \otimes \downarrow_H^G\uparrow_H^G W.$$

But we have a map $j\colon W \longrightarrow \downarrow_H^G\uparrow_H^G(W)$ (the adjoint of the identity map of $\uparrow_H^G(W)$) given by inclusion at the identity coset of H in G. So then we get an H-equivariant map

$$\mathrm{id}\otimes j\colon\ \downarrow_H^G(V)\otimes W \longrightarrow \downarrow_H^G(V) \otimes \downarrow_H^G\uparrow_H^G W$$

as desired. It remains to show that the corresponding G-equivariant map is an isomorphism, and this means simply that it is an isomorphism of underlying vector spaces. But at the level of vector spaces, this map is the distributivity isomorphism

$$\bigoplus_{G/H}(V\otimes W) \cong V \otimes \bigoplus_{G/H} W$$

for direct sums over a tensor product. ∎

Example 1.1.43. We'll use the projection formula to complete the argument for the isomorphism type of $\lambda_4 \otimes \lambda_4$ in $RO(C_4)$ (Example 1.1.34). Recall from Example 1.1.38 that $\uparrow_{C_2}^{C_4}(\sigma) = \lambda_4$. So in the projection formula, taking $V = \lambda_4$ and $W = \sigma$, we get

$$\lambda_4 \otimes \uparrow_{C_2}^{C_4}(\sigma) = \uparrow_{C_2}^{C_4}\Big(\downarrow_{C_2}^{C_4}(\lambda_4)\otimes\sigma\Big),$$

which gives

$$\lambda_4 \otimes \lambda_4 = \uparrow_{C_2}^{C_4}(2\sigma\otimes\sigma) = \uparrow_{C_2}^{C_4}(\mathbf{2}) = \mathbf{2} \oplus 2\sigma.$$

1.1.7. *Coinduction is induction.* Last time, we saw in Proposition 1.1.40 that induction is left adjoint to restriction. In fact, induction is also right adjoint to restriction (this is very unusual). We will see this by first showing that restriction has a right adjoint, and then identifying the right adjoint with induction.

Definition 1.1.44. Let $H \le G$. We define **coinduction** $\mathrm{Coind}_H^G\colon \mathbf{Mod}_{\mathbf{k}[H]} \longrightarrow \mathbf{Mod}_{\mathbf{k}[G]}$ by the formula

$$\mathrm{Coind}_H^G(V) = \mathrm{Hom}_{\mathbf{k}[H]}(\mathbf{k}[G], V).$$

Here, we must be a little careful to specify what we mean. First, V starts off as a left $\mathbf{k}[H]$-module, and we can consider $\mathbf{k}[G]$ also as a left $\mathbf{k}[H]$-module. Then we take the vector space of $\mathbf{k}[H]$-module maps, a.k.a. H-equivariant maps. But we want the result to be a (left) $\mathbf{k}[G]$-module, and this action arises as follows. We may consider $\mathbf{k}[G]$ as a right $\mathbf{k}[G]$-module. Then the set $\mathrm{Hom}_{\mathbf{k}[H]}(\mathbf{k}[G], V)$ acquires a left $\mathbf{k}[G]$-module structure by the formula

$$g \cdot f\left(\sum_{h\in G} c_h h\right) = f\left(\sum_{h\in G} c_h h \cdot g\right).$$

Watch the video (in canvas): the **left** G-action on $\mathrm{Coind}_H^G(V)$

Like Proposition 1.1.40, the following result is a special case of a statement about modules.

Proposition 1.1.45. *Let $H \leq G$. Then coinduction is right adjoint to restriction:*

$$\downarrow_H^G : \mathbf{Mod}_{\mathbf{k}[G]} \rightleftarrows \mathbf{Mod}_{\mathbf{k}[H]} : \mathrm{Coind}_H^G.$$

Combining Proposition 1.1.40 and Proposition 1.1.45, we learn that restriction is *both* a left and right adjoint, which means that it will preserve almost any construction (for example, limits and colimits) you might like. Moreover, we have

Proposition 1.1.46. *Let $H \leq G$. Then there is a natural isomorphism of functors $\uparrow_H^G(-) \cong \mathrm{Coind}_H^G$.*

This is an important result that has an analogue in the world of equivariant stable homotopy theory (it goes by the name of the Wirthmuller isomorphism in that context). Applying Proposition 1.1.46 to the case $V = \mathbf{1}$ gives an isomorphism of G-representations

$$\mathbf{k}[G/H] = \uparrow_H^G(\mathbf{1}) \cong \mathrm{Coind}_H^G(\mathbf{1}) \cong \mathrm{Hom}_{\mathbf{k}[H]}(\mathbf{k}[G], \mathbf{1}) \cong \mathrm{Hom}_{\mathbf{k}}(\mathbf{k}[H\backslash G], \mathbf{1}).$$

This is sometimes summarized as the statement that *orbits are self-dual*.

Proof. For any H-representation V, we want to produce an isomorphism

$$\alpha_V : \uparrow_H^G(V) = \mathbf{k}[G] \otimes_{\mathbf{k}[H]} V \cong \mathrm{Hom}_{\mathbf{k}[H]}(\mathbf{k}[G], V) = \mathrm{Coind}_H^G(V)$$

of G-representations. Let us first look at induction. Here, we use the right H-action on $\mathbf{k}[G]$. Let $\{g_i H\}$ be a set of representatives for the decomposition of G into left cosets. Note that each left coset $g_i H$ is a free right H-set. Then

$$\uparrow_H^G(V) = \mathbf{k}[G] \otimes_{\mathbf{k}[H]} V \cong \left(\bigoplus_{g_i H \in G/H} \mathbf{k}[g_i H]\right) \otimes_{\mathbf{k}[H]} V \cong \bigoplus_{g_i H \in G/H} V.$$

On the other hand, for coinduction, we are using the left H-action on $\mathbf{k}[G]$. Using the same choice of representatives, but now using them to represent right cosets, we have

$$\mathrm{Coind}_H^G(V) = \mathrm{Hom}_{\mathbf{k}[H]}(\mathbf{k}[G], V) \cong \mathrm{Hom}_{\mathbf{k}[H]}\left(\bigoplus_{Hg_i \in H\backslash G} \mathbf{k}[Hg_i], V\right) \cong \prod_{Hg_i \in H\backslash G} V.$$

Now define α_V to be the map that takes the copy of V labeled by the left coset $g_i H$ identically onto the factor V labeled by the right coset Hg_i^{-1}. Then α_V is an isomorphism, and it is G-equivariant. In other words, α_V is an isomorphism of G-representations. ∎

Although all of the discussion in this section is for $\mathbf{k}[G]$-modules, it applies equally well in the more general context of $R[G]$-modules.

$\boxed{\textbf{Wed, Sept. 2}}$

1.1.8. *The double coset formula.* We have recently discussed restriction and induction of representations. The double coset formula describes what happens when we combine these two operations. The general setup will be an ambient group G and a pair of subgroups H and K in G. We will describe the composition

$$\begin{tikzcd} & RO(G) \arrow[dr, "\mathrm{Res}"] & \\ RO(H) \arrow[ur, "\mathrm{Ind}"] & & RO(K). \end{tikzcd}$$

Visually, the answer is that this composition will factor as a sum of compositions of the following form, where we write K^g for the conjugate subgroup gKg^{-1} and c_g for conjugation by g:

$$\begin{tikzcd} RO(H) \arrow[dr, "\mathrm{Res}"] & & & RO(K). \\ & RO(H \cap K^g) \arrow[r, "\cong", "c_g^*"'] & RO(H^{g^{-1}} \cap K) \arrow[ur, "\mathrm{Ind}"] & \end{tikzcd}$$

In order to state the result, we need the language of double cosets.

Definition 1.1.47. For subgroups H and K in G, we can think of H as acting on the left of G and K acting on the right of G. The quotient with respect to both of these actions is the set $H\backslash G/K$ of **double cosets** of H and K in G.

Thus a typical double coset is

$$HgK = \{x \in G \mid x = hgk \text{ for some } h \in H \text{ and } k \in K\}.$$

Theorem 1.1.48 (Mackey's double coset formula)**.** *For subgroups H and K of G and an H-representation V, we have a decomposition of K-representations*

(Double coset formula) $$\downarrow_K^G \uparrow_H^G (V) \cong \sum_{HgK \in H\backslash G/K} \uparrow_{H^{g^{-1}} \cap K}^K \Big(c_g^* \big(\downarrow_{H \cap K^g}^H (V) \big) \Big)$$

This formula looks intimidating. Personally, I think of the visual description given above Definition 1.1.47 and then recover the formula if I need it. We omit the proof. It is not terribly difficult; it essentially amounts to carefully analyzing how the decomposition that is already present in the definition of induced representations interacts with the K-action.

Corollary 1.1.49. *Suppose that G is abelian and $H = K$. Then the composition*

$$RO(H) \xrightarrow{\uparrow_H^G} RO(G) \xrightarrow{\downarrow_H^G} RO(H)$$

is multiplication by the index $|G/H|$ of H in G.

Example 1.1.50. Take $H = C_2$ inside of $G = C_4$. The composition

$$RO(C_2) \xrightarrow{\uparrow_{C_2}^{C_4}} RO(C_4) \xrightarrow{\downarrow_{C_2}^{C_4}} RO(C_2)$$

is given by

$$\mathbf{1} \mapsto \mathbf{1} \oplus \sigma \mapsto \mathbf{2},$$

$$\sigma \mapsto \lambda_4 \mapsto 2\sigma.$$

Example 1.1.51. In the video on induction and restriction in D_4, you can check that in most cases, inducing up and restricting back down simply multiplies by 2. One case where this is not true is the composition

$$RO(K_4) \xrightarrow{\uparrow_{K_4}^{D_4}} RO(D_4) \xrightarrow{\downarrow_{K_4}^{D_4}} RO(K_4), \tag{1.1.52}$$

where K_4 is generated by r^2 and s. As D_4 is not abelian, Corollary 1.1.49 does not apply. We therefore apply the full Theorem 1.1.48. Now there are two right cosets of K_4 in D_4, namely K_4e and K_4r. The right action of K_4 on $K_4\backslash D_4$ is trivial, so we have the two double cosets K_4eK_4 and K_4rK_4.

For the identity coset, the restriction-conjugation-induction composite in Mackey's formula is the identity map of $RO(K)$. For the coset K_4rK_4, we note that K_4 is normal, and so the restriction and induction maps are the identity. However, conjugation by r is *not* the identity on K_4: it interchanges the elements s and sr^2. The effect on the representation ring is that c_r interchanges the sign representations σ_{r^2} and $\sigma_{r^2} \otimes \sigma_s$. It follows that the composition (1.1.52) is given by

$$\begin{gathered}
\mathbf{1} \mapsto \mathbf{1} \oplus \mathbf{1} = \mathbf{2}, \\
\sigma_{r^2} \mapsto \sigma_{r^2} \oplus (\sigma_{r^2} \otimes \sigma_s), \\
\sigma_s \mapsto \sigma_s \oplus \sigma_s = 2\sigma_s, \quad \text{and} \\
\sigma_{r^2} \otimes \sigma_s \mapsto (\sigma_{r^2} \otimes \sigma_s) \oplus \sigma_s.
\end{gathered}$$

This agrees with the matrices given in the D_4 video.

1.1.9. *Fixed points.*

Definition 1.1.53. Given a G-representation, or more generally an $R[G]$-module M for some commutative ring R, we define the **fixed points** to be

$$M^G = \{m \in M \mid g \cdot m = m \text{ for all } g \in G\}.$$

For a G-representation V, the fixed points $V^G \subset V$ is the largest trivial subrepresentation. Notice also that for any G-representation V, a G-equivariant map from a trivial representation to V must land in the G-fixed points. Even better, we have

Proposition 1.1.54. *The fixed points functor is right adjoint to the trivial G-action functor*

$$\mathbf{triv} = q^* \colon \mathbf{Mod}_R \rightleftarrows \mathbf{Mod}_{R[G]} \colon (-)^G.$$

We can also take H-fixed points for a subgroup H. We define those by first restricting to H and then passing to fixed points.

Definition 1.1.55. Given a G-representation, or more generally an $R[G]$-module M for some commutative ring R, we define the H**-fixed points** functor as the composite

$$\mathbf{Mod}_{R[G]} \xrightarrow{\downarrow_H^G} \mathbf{Mod}_{R[H]} \xrightarrow{(-)^H} \mathbf{Mod}_R.$$

Example 1.1.56. Take $G = K_4$. Then the L-fixed points homomorphism is given by

$$\begin{gathered}
\mathbf{1} \mapsto \mathbf{1} \\
p_1^*(\sigma) \mapsto \mathbf{0} \\
m^*(\sigma) \mapsto \mathbf{0} \\
p_2^*(\sigma) \mapsto \mathbf{1}.
\end{gathered}$$

Mon, Sept. 7

1.2. **Mackey functors.** One of the concepts we've been building towards is the notion of Mackey functor. The reason is that, when we get to *equivariant stable homotopy theory* later in the course, Mackey functors there will play the role that abelian groups play in ordinary stable homotopy theory. For more complete treatments of Mackey Functors, see [W, TW].

Definition 1.2.1. A **Mackey functor** $\underline{M}$ for the group G consists of the following data:

- an abelian group $\underline{M}(H)$ for each subgroup $H \leq G$
- a "restriction" homomorphism $R^K_H \colon \underline{M}(K) \longrightarrow \underline{M}(H)$ whenever $H \leq K$
- a "transfer" or "induction" homomorphism $I^K_H \colon \underline{M}(H) \longrightarrow \underline{M}(K)$ whenever $H \leq K$ (notice this goes in the other direction)
- a conjugation homomorphism $c_g \colon \underline{M}(H) \longrightarrow \underline{M}(H^g)$ for each $H \leq G$ and $g \in G$

This should remind you of the maps we had between $RO(H)$ and $RO(G)$. These are subject to the following conditions, which are precisely the properties satisfied in the case of representation rings.

(1) R^H_H and I^H_H are the identity of $\underline{M}(H)$. Moreover, for each $h \in H$, c_h is the identity of $\underline{M}(H)$.
(2) $R^K_L \circ R^H_K = R^H_L$ and $I^H_K \circ I^K_L = I^H_L$ whenever $L \leq K \leq H$
(3) $c_g \circ c_h = c_{gh}$ for all g and h in G
(4) $R^{H^g}_{K^g} c_g = c_g R^H_K$ and $I^{H^g}_{K^g} c_g = c_g I^H_K$ for all $K \leq H$ and g in G
(5) The double coset formula holds, meaning that

$$R^H_L \circ I^H_K = \sum_{KhL \in K\backslash H/L} I^L_{K^{h^{-1}} \cap L}\, c_{h^{-1}}\, R^K_{K \cap L^h}$$

for all $L, K \leq H \leq G$.

Again, representation rings give an example of such a structure.

Example 1.2.2. For any G, let $\underline{RO}_G(H)$ be the group $RO(H)$. We have already defined restriction and induction homomorphisms. For the conjugation, there is a slight subtlety in that conjugation by g is an isomorphism $H \longrightarrow H^g$, so that pullback along this map is what was previously denoted $(c_g)^* \colon RO(H^g) \longrightarrow RO(H)$. This goes in the opposite direction than what is asked for in the definition of a Mackey functor, so we just take the inverse and define $c_g \colon RO(H) \longrightarrow RO(H^g)$ to be $c_g = (c_{g^{-1}})^*$. We have previously done the work to establish that $\underline{RO}_G$ is a Mackey functor.

In the video on Restriction and Induction in D_4, we are really describing the Mackey functor $\underline{RO}_{D_4}$.

Remark 1.2.3. Let $\underline{M}$ be a G-Mackey functor. If g normalizes H, meaning that $H^g = H$, then c_g maps $\underline{M}(H)$ to itself. In other words, the normalizer $N_G(H)$ of H in G acts on $\underline{M}$. Moreover, the (normal) subgroup $H \leq N_G(H)$ acts trivially, meaning that we get an induced action of the quotient group $N_G(H)/H$. Recall that this group is called the **Weyl group** of H in G and is denoted $W_G(H)$. Summing up, we have an action of $W_G(H)$ on $\underline{M}(H)$.

An extreme case of this is when $H = e$. Then the Weyl group is G itself, and we conclude that we get a G-action on $\underline{M}(e)$.

Note that when G is abelian, then $N_G(H) = G$ for any subgroup H, and so the Weyl group $W_G(H)$ is the quotient group G/H.

Watch the video (in canvas): Restriction, transfers, and Weyls!

In the case of the Mackey functor $\underline{RO}_G$, the Weyl group action at $\underline{RO}_G(e) = RO(e) \cong \mathbb{Z}$ is trivial. But at a nontrivial subgroup, the Weyl group action can be nontrivial. We saw this in

Example 1.1.51, where conjugation by r on the subgroup $K_4 = \langle r^2, s\rangle$ induced a nontrivial automorphism of $RO(K_4)$.

Definition 1.2.4. Let M be an $R[G]$-module. We define a Mackey functor $\underline{F(M)}$ by $\underline{F(M)}(H) = M^H$, the H-fixed points of M. If $H \leq K$, then the restriction map $R_H^K\colon M^K \longrightarrow M^H$ is simply the inclusion of fixed points for a larger subgroup. The transfer map is defined by

$$I_H^K\colon M^H \longrightarrow M^K, \qquad I_H^K(m) = \sum_{kH \in K/H} k \cdot m.$$

This does not depend on coset representatives, since m is assumed to be fixed by H. Moreover, the sum is fixed by K since multiplying by any element of K will simply permute the coset representatives. Finally, the homomorphism $c_g\colon M^H \longrightarrow M^{H^g}$ is given by acting by the element g. It is then straight-forward to verify the Mackey functor axioms, save for the double coset formula.

Example 1.2.5. An important example of the construction given in Definition 1.2.4 is when M starts as a trivial G-module. For an abelian group A with trivial G-action, the resulting Mackey functor is known as the **constant Mackey functor** $\underline{A}$ at A. All of the restriction maps and conjugation maps are the identity, while the transfer $I_H^K\colon \underline{A}(H) \longrightarrow \underline{A}(K)$ is multiplication by the index of H in K.

Example 1.2.6. Take M to be the free module $\mathbb{Z}[C_2]$. Then we can display the Mackey functor $\underline{F(\mathbb{Z}[C_2])}$ in the following form, which is sometimes called a "Lewis diagram", after Gaunce Lewis:

$$\underline{F(\mathbb{Z}[C_2])} = \begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "\Delta"'] \\ \mathbb{Z}[C_2] \arrow[u, bend right, "\nabla"'] \arrow[loop below] \end{tikzcd}$$

Here we display $[\underline{F(\mathbb{Z}[C_2])}(C_2)$ at the top of the diagram and $[\underline{F(\mathbb{Z}[C_2])}(e)$ at the bottom. The map pointing downwards is the restriction. We have labeled it Δ, for a diagonal map. That implicitly means we have a basis in mind for $\mathbb{Z}[C_2]$, and we take the group elements of C_2 as our basis, as usual. The map pointing up is the transfer, and we write ∇ for the "fold" map, which sends both of our basis elements to 1. We have drawn in an arrow on the bottom node as well to remind us that there is a Weyl group action. In this case, the action is swapping the two basis elements e and τ in $\mathbb{Z}[C_2]$.

By a map of G-Mackey functors $\underline{M} \longrightarrow \underline{N}$ (for fixed G), we simply mean a collection of homomorphisms $\underline{M}(H) \longrightarrow \underline{N}(H)$, as H runs over the subgroups of G, which are compatible with the restriction, transfer, and conjugation homomorphisms. We thus get a category of Mackey functors for G, which we denote by $\mathbf{Mack}(G)$. As we have already said, evaluation at the trivial subgroup gives

$$\mathbf{ev}_e\colon \mathbf{Mack}(G) \longrightarrow \mathbf{Mod}_{\mathbb{Z}[G]}.$$

In fact, we have

Proposition 1.2.7. *For each G, the evaulation functor $\mathbf{ev}_e$ is left adjoint to the fixed point functor:*

$$\mathbf{ev}_e\colon \mathbf{Mack}(G) \rightleftarrows \mathbf{Mod}_{\mathbb{Z}[G]}\colon F.$$

We include the proof below, but feel free to skip it on first reading.

Proof. Let $\underline{M}$ be a G-Mackey functor and N a $\mathbb{Z}[G]$-module. We wish to show that maps of Mackey functors $\underline{M} \longrightarrow \underline{F(N)}$ correspond to maps of G-modules $\underline{M}(e) \longrightarrow N$. Since $\underline{F}(N)(e) = N$, evaluating a map of Mackey functors $\underline{M} \longrightarrow \underline{F(N)}$ at the trivial subgroup gives a G-module map $\underline{M}(e) \longrightarrow N$ (the equivariance is guaranteed since a map of Mackey functors must be compatible with the Weyl-group actions).

On the other hand, given a G-equivariant map $\underline{M}(e) \longrightarrow N$ and a subgroup $H \leq G$, in order to obtain a map of Mackey functors $\underline{M} \longrightarrow \underline{F(N)}$, we need to know that the composition

$$\underline{M}(H) \xrightarrow{R_e^H} \underline{M}(e) \longrightarrow N$$

factors through N^H. But recall that in $\underline{M}$, the homomorphism c_h is the identity on $\underline{M}(H)$, and we have $c_h \circ R_e^H = R_e^H \circ c_h = R_e^H$. In other words, R_e^H has image in the H-fixed points of $\underline{M}(e)$. It follows that we get a commuting square

$$\begin{tikzcd} \underline{M}(H) \arrow[r] \arrow[d, "R_e^H"'] & N^H \arrow[d] \\ \underline{M}(e) \arrow[r] & N. \end{tikzcd}$$

For the transfer maps, we want to know that

$$\begin{tikzcd} \underline{M}(H) \arrow[r] & N^H \\ \underline{M}(e) \arrow[r] \arrow[u, "I_e^H"'] & N \arrow[u] \end{tikzcd}$$

commutes. We are comparing two maps to N^H. Since the inclusion $N^H \hookrightarrow N$ is injective, it suffices to show that the two maps $\underline{M}(e) \rightrightarrows N^H$ agree after including into N. But the up-across-down composition agrees with up-down-across, since we already know that our map is compatible with restrictions. And the double coset formula tells us that up-down in $\underline{M}$ is a sum over $h \in H$ of the action by h. Since this is also what is given by across-up-down, we have shown that the map is compatible with transfer maps. We leave as an exercise to check that the same squares commute for any pair $H \leq K$ of subgroups. And we also leave as an exercise the verification that the map is compatible with the conjugation homomorphisms. ■

Wed, Sept. 9

Last time, we introduced the idea of a Mackey functor (Definition 1.2.1) for a group G. We also introduced the fixed-point functor which produces a Mackey functor from the diagram of fixed points of a G-module. We saw that this construction was right adjoint to the functor that evaluates a Mackey functor at the trivial subgroup. There is also an orbit construction:

Definition 1.2.8. Let M be an $R[G]$-module. We define a Mackey functor $\underline{Q(M)}$ by $\underline{Q(M)}(H) = M/H$, the orbits under the H-action. As usual, we are being a little sloppy here in writing M/H even though we are passing to a quotient from a *left* H-action. More honest notation would be $H\backslash M$.

If $H \leq K$, then the restriction map $R^K_H \colon M/K \longrightarrow M/H$ is given by

$$R^K_H(\overline{m}) = \sum_{kH \in K/H} \overline{k \cdot m}.$$

The transfer map $I^K_H \colon M/H \longrightarrow M/K$ is simply the quotient map. Finally, the conjugation map $c_g \colon M/H \longrightarrow M/H^g$ is given by $c_g(\overline{m}) = \overline{g \cdot m}$. Again, we leave it as an exercise to verify the Mackey axioms.

Example 1.2.9. In parallel to Example 1.2.5, an important example of the quotient Mackey functor construction is when we start with a trivial G-module. Thus let A be an abelian group, equipped with a trivial action of G. Then the quotient Mackey functor $\underline{Q(A)}$ is known as the **dual constant Mackey functor**, written $\underline{A}^*$. Here, the transfer and conjugation maps are the identity, while the restriction maps are multiplication by the index of H in K.

Similarly to Proposition 1.2.7, we have

Proposition 1.2.10. *For each G, the evaulation functor* $\mathbf{ev}_e$ *is right adjoint to the quotient functor:*

$$Q \colon \mathbf{Mod}_{\mathbb{Z}[G]} \rightleftarrows \mathbf{Mack}(G) \colon \mathbf{ev}_e.$$

This result tells us that the functor Q will take free $\mathbb{Z}[G]$-modules to "free" Mackey functors. More precisely, we have

$$\begin{aligned} \text{Maps of Mackey functors } \underline{Q(\mathbb{Z}[G])} \longrightarrow \underline{M} &\leftrightarrow \text{Maps of } \mathbb{Z}[G]\text{-modules } \mathbb{Z}[G] \longrightarrow \underline{M}(e) \\ &\leftrightarrow \text{Maps of } \mathbb{Z}\text{-modules } \ \mathbb{Z} \longrightarrow \underline{M}(e) \\ &\leftrightarrow \text{elements of } \underline{M}(e) \end{aligned}$$

Example 1.2.11. As we explain in the video "A short exact sequence of C_2-Mackey functors", the C_2-Mackey functor $Q(\mathbb{Z}[C_2])$ agrees with $F(\mathbb{Z}[C_2])$ of Example 1.2.6.

Definition 1.2.12. For a finite group G, the **Burnside ring** of G, denoted $A(G)$, is the quotient

$$A(G) = \mathbb{Z}\left\{\text{isomorphism classes of finite } G\text{-sets}\right\} / \langle [X \amalg Y] - [X] - [Y] \rangle.$$

Since finite G-sets decompose into a disjoint union of orbits and the isomorphism type of an orbit G/H corresponds to the conjugacy class of the subgroup H, we can get away with this simpler definition: writing $\mathrm{Conj}(G)$ for the set of conjugacy classes of subgroups in G, we have

$$A(G) \cong \bigoplus_{\mathrm{Conj(G)}} \mathbb{Z}.$$

We have a "linearization" homomorphism $A(G) \longrightarrow RO(G)$ which takes a G-set X to the permutation representation $\mathbb{R}[X]$. Our previous discussions imply that this is a ring homomorphism. Like Example 1.2.2, we can assemble the Burnside rings for the subgroups of G into a Mackey functor.

Example 1.2.13. For any G, let $\underline{A}_G(H)$ be the group $A(H)$. The restriction map $R^K_H\colon A(K) \longrightarrow A(H)$ takes a set X equipped with an action of K and simply restricts the action to the subgroup H. The transfer map $I^K_H\colon A(H) \longrightarrow A(K)$ takes a set X equipped with an action of H and produces the K-set $K \times_H X$. The conjugation map $c_g\colon A(H) \longrightarrow A(H^g)$ takes an H-set X defined by a homomorphism $H \longrightarrow \mathrm{Aut}(X)$ to the H^g-set defined by $H^g \xrightarrow{conj_{g^{-1}}} H \longrightarrow \mathrm{Aut}(X)$.

It turns out that the Mackey functor $\underline{A}_G$ is also free. In order to give a precise formulation of this statement, we point out that if A is an abelian group and $\underline{M}$ is a Mackey functor, then we can form a Mackey functor $A \otimes \underline{M}$ by simply taking the tensor product $A \otimes \underline{M}(H)$ at each subgroup H.

Proposition 1.2.14. *For each G, the evaulation functor $\mathbf{ev}_G$ is right adjoint to tensoring with $\underline{A}_G$:*

$$- \otimes \underline{A}_G\colon \mathbf{Mod}_{\mathbb{Z}} \rightleftarrows \mathbf{Mack}(G)\colon \mathbf{ev}_G.$$

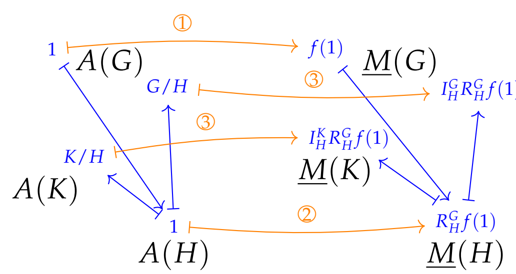
As a result, we have

$$\begin{aligned}\text{Maps of Mackey functors } \mathbb{Z} \otimes \underline{A}_G \cong \underline{A}_G \longrightarrow \underline{M} &\leftrightarrow \text{Maps of } \mathbb{Z}\text{-modules } \mathbb{Z} \longrightarrow \underline{M}(G) \\ &\leftrightarrow \text{elements of } \underline{M}(G)\end{aligned}$$

Sketch of proof. We discuss why an element of $\underline{M}(G)$, or equivalently a homomorphism $f\colon \mathbb{Z} \longrightarrow \underline{M}(G)$, gives rise to a map of Mackey functors $\underline{A}_G \longrightarrow \underline{M}$. Given such a homomorphism f, we take

$$\mathbb{Z} \hookrightarrow A(G) \longrightarrow \underline{M}(G)$$

to be f. Since $1 \in A(G)$ restricts to $1 \in A(H) = \underline{A}_G(H)$ for each H, a map of Mackey functors must necessarily take $1 \in A(H)$ to $R^G_H(f(1))$. Now $G/H \in A(G)$ is $I^G_H(1)$, so a map of Mackey functors must take $G/H \in A(G)$ to $I^G_H R^G_H(1) \in \underline{M}(G)$. Similarly, $K/H \in \underline{A}_G(K) = A(K)$ must be sent to $I^K_H R^G_H(f(1)) \in \underline{M}(K)$. ∎



Example 1.2.15. The element $1 \in \mathbb{Z} = Q(\mathbb{Z}[C_2])(C_2)$ determines the map of Mackey functors $\underline{A}_{C_2} \longrightarrow F(\mathbb{Z}[C_2])$

$$\begin{tikzcd}
A(C_2) \cong \mathbb{Z}\{1, C_2\} \arrow[r, "(1\ \ 2)"] \arrow[d, bend right, "(1\ \ 2)"'] & \mathbb{Z} \arrow[d, bend right, "\Delta"'] \\
\mathbb{Z} \arrow[r, "\Delta"] \arrow[u, bend right, "\binom{0}{1}"'] & \mathbb{Z}[C_2] \arrow[u, bend right, "\nabla"']
\end{tikzcd}$$

Like the category $\mathbf{Mod}_{R[G]}$ of G-modules, the category $\mathbf{Mack}(G)$ of G-Mackey functors is a good place for doing (homological) algebra. More precisely, it is an abelian category. First of all, this means that we can form direct sums of Mackey functors. The direct sum $\underline{M} \oplus \underline{N}$ is given at the subgroup H by the sum of abelian groups $\underline{M}(H) \oplus \underline{N}(H)$, and the restriction, transfer, and conjugation maps are given coordinate-wise. Moreover, we can make sense of the kernel of a map of Mackey functors. Again, everything is just defined levelwise. And we can similarly make sense of a cokernel of a map of Mackey functors, again defined as the levelwise quotient.

Watch the video (in canvas): A short exact sequence of C_2-Mackey functors

$\boxed{\textbf{Mon, Sept. 14}}$

In the video from last class, we saw that the short exact sequence

$$\mathbb{Z} \hookrightarrow \mathbb{Z}[C_2] \twoheadrightarrow \mathbb{Z}_{\mathrm{sgn}}$$

produces a short exact sequence of Mackey functors

$$\underline{\mathbb{Z}}^* = Q(\mathbb{Z}) \hookrightarrow Q(\mathbb{Z}[C_2]) \twoheadrightarrow Q(\mathbb{Z}_{\mathrm{sgn}}).$$

This might lead you to believe that the functor $Q\colon \mathbf{Mod}_{\mathbb{Z}[C_2]} \longrightarrow \mathbf{Mack}C_2$ is exact, meaning that it sends exact sequences to exact sequences. However, the following example shows that this is not true in general.

Example 1.2.16. We also have a (nonsplit) short exact sequence of $\mathbb{Z}[C_2]$-modules

$$\mathbb{Z}_{\mathrm{sgn}} \hookrightarrow \mathbb{Z}[C_2] \twoheadrightarrow \mathbb{Z},$$

sending $1 \in \mathbb{Z}_{\mathrm{sgn}}$ to $1-\tau \in \mathbb{Z}[C_2]$. Applying the quotient Mackey functor to this sequence produces the maps of Mackey functors

$$\begin{tikzcd}
\mathbb{F}_2 \arrow[r, "0"] & \mathbb{Z} \arrow[r, "1"] \arrow[d, bend right, "\Delta"'] & \mathbb{Z} \arrow[d, bend right, "2"'] \\
\mathbb{Z}_{\mathrm{sgn}} \arrow[u, two heads] \arrow[r, hook, "1-\tau"] & \mathbb{Z}[C_2] \arrow[r, two heads, "\nabla"] \arrow[u, bend right, "\nabla"'] & \mathbb{Z}. \arrow[u, bend right, "1"']
\end{tikzcd}$$

In this case, the map on orbits $\mathbb{F}_2 \longrightarrow \mathbb{Z}$ is not injective.

We mentioned last time that the category $\mathbf{Mack}(G)$ of G-Mackey functors is an abelian category. A useful notion in the context of abelian categories is that of a *projective* object. Recall that an object X in an abelian category $\mathcal{C}$ is projective if $\mathrm{Hom}(X,-)\colon \mathcal{C} \longrightarrow \mathbf{Ab}$ is exact. This is equivalent to the condition that $\mathrm{Hom}(X,-)$ preserves epimorphisms.

Proposition 1.2.17. *For any group G, the Mackey functors $Q(\mathbb{Z}[G])$ and $\underline{A}_G$ are projective Mackey functors.*

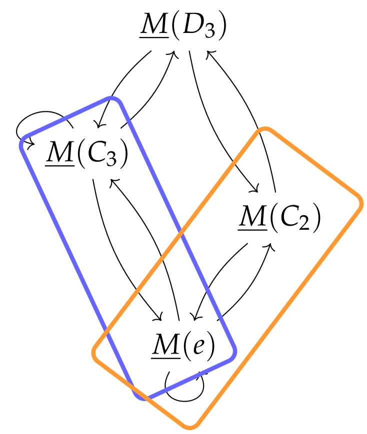
Proof. An epimorphism $\underline{M} \longrightarrow \underline{N}$ of G-Mackey functors is simply a morphism such that $\underline{M}(H) \longrightarrow \underline{N}(H)$ is surjective for each H. The result now follows from Proposition 1.2.10 and Proposition 1.2.14. For the Mackey functor $\underline{A}_G$, we know that $\mathrm{Hom}(\underline{A}_G, \underline{M}) \cong \underline{M}(G)$ for any $\underline{M} \in \mathbf{Mack}(G)$. Thus if $\underline{M} \longrightarrow \underline{N}$ is an epimorphism, we deduce that $\mathrm{Hom}(\underline{A}_G, \underline{M}) \longrightarrow \mathrm{Hom}(\underline{A}_G, \underline{N})$ is an epimorphism (surjection) as well. ∎

Proposition 1.2.17 can also be viewed as an application of the more general principle that if $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ is an adjoint pair between abelian categories such that R preserves epimorphisms, then L will preserve projective objects.

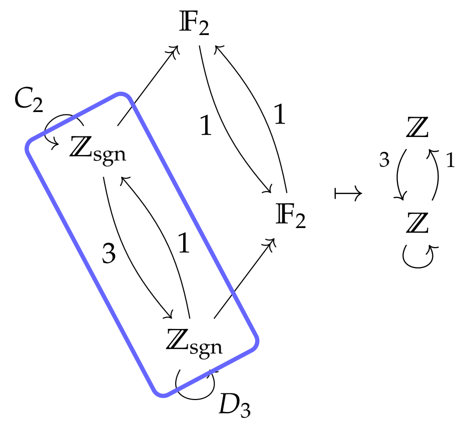
1.2.1. *Restriction and induction.* We have found left adjoints to evaluation at the trivial subgroup e and the whole G. We can also describe a left adjoint to evaluation at an arbitrary subgroup $H \leq G$. For this, it is useful to notice that evaluation at H can be thought of as first restricting your Mackey functor to an H-Mackey functor, and then evaluating that Mackey functor at the (whole) group H. For this reason, we now discuss restriction of Mackey functors, and the left adjoint induction. We start with restriction, which is straightforward.

Definition 1.2.18. Let $\underline{M} \in \mathbf{Mack}(G)$ and $H \leq G$. We define the **restriction** $\downarrow_H^G \underline{M} \in \mathbf{Mack}(H)$ by $\downarrow_H^G \underline{M}(K) = \underline{M}(K)$ whenever $K \leq H$. The restriction, transfer, and conjugation maps are those of the G-Mackey functor $\underline{M}$.

Example 1.2.19. To the right, we display a D_3-Mackey functor $\underline{M}$. In blue, we have displayed the restriction of $\underline{M}$ to C_3, and the restriction of $\underline{M}$ to a cycle 2-subgroup is displayed in orange. One thing to be careful of is that we have fewer available conjugation maps once we restrict, which means that the Weyl groups become smaller. For instance, $W_{D_3}(C_3) \cong D_3/C_3 \cong C_2$, whereas $W_{C_3}(C_3)$ is trivial.



Example 1.2.20. Consider the D_3-Mackey functor $Q(\mathbb{Z}_{\mathrm{sgn}})$. The restriction to the subgroup C_3 is $\downarrow^{D_3}_{C_3}(Q(\mathbb{Z}_{\mathrm{sgn}})) \cong \underline{\mathbb{Z}}^*$, as shown below:



Just as we saw for representations, restriction has a left adjoint, induction. The simplest way to define it is to first give an alternative description of Mackey functors. We have indexed the values $\underline{M}(H)$ on the subgroups of G. We can equally well think of this as indexed by the finite G-set G/H. Since any finite G-set decomposes as a disjoint union of orbits, we can then define $\underline{M}(X)$ for any finite G-set by first decomposing X into orbits and then declaring $\underline{M}(X)$ to be the direct sum of the values $\underline{M}(G/H)$ for the orbits appearing in X.[1]

Definition 1.2.21. Let $\underline{M} \in \mathbf{Mack}(H)$, thought of as indexed over finite H-sets, and let $H \leq G$. We define the **induction** $\uparrow^G_H \underline{M}$ of $\underline{M}$ from H to G by setting

$$\uparrow^G_H \underline{M}(X) = \underline{M}(\downarrow^G_H X),$$

where here we denote by $\downarrow^G_H X$ the restriction of the G-action on X down to the H-action. The restriction, transfer, and conjugation homomorphisms are also given by this formula.

Watch the video (in canvas): The induced C_2-Mackey functor $\uparrow^{C_2}_e \underline{\mathbb{Z}} \cong Q\mathbb{Z}[C_2]$

[1] There is one subtlety here in that although any finite G-set does decompose canonically into a disjoint union of orbits, there is no canonical identification of an orbit with a G-set of the form G/H. Such an identification amounts to choosing a point on the orbit, and of course there is no canonical such choice.

Wed, Sept. 16

Last time, we gave a definition of induced Mackey functors.

Remark 1.2.22. It is also possible to give a formula for induction in the indexing-over-subgroups formulation of Mackey functors. The formula is

$$\uparrow_H^G \underline{M}(K) = \bigoplus_{HgK \in H\backslash G/K} \underline{M}(H \cap K^g).$$

Example 1.2.23. Recall that we write $L \leq K$ for the subgroup $C_2 \times e \leq C_2 \times C_2$. Consider the induction functor $\uparrow_L^K$. Thus let $\underline{M} \in \mathbf{Mack}(L) = \mathbf{Mack}(C_2)$. Then the induced K-Mackey functor $\uparrow_L^K \underline{M}$ is

$$\begin{tikzcd}
 & & \underline{M}(L) \arrow[dll, bend right, "\Delta"'] \arrow[d, bend right] \arrow[drr, bend left] & & \\
\mathbb{Z}[K/L] \otimes \underline{M}(L) \arrow[urr, bend right, "\nabla"'] \arrow[drr, bend right] & & \underline{M}(e) \arrow[u, bend right] \arrow[d, bend right, "\Delta"'] & & \underline{M}(e) \arrow[ull, bend left] \arrow[dll, bend right, "\Delta"'] \\
 & & \mathbb{Z}[K] \otimes_{\mathbb{Z}[L]} \underline{M}(e) \arrow[ull, bend right] \arrow[u, bend right, "\nabla"'] \arrow[urr, bend right, "\nabla"'] \arrow[loop below] & &
\end{tikzcd}$$

Here, to describe the restriction and transfer between L and the trivial subgroup, it is helpful to rewrite the value at L as $\mathbb{Z}[K] \otimes_{\mathbb{Z}[L]} \underline{M}(L)$, where we think of L as acting trivially on $\underline{M}(L)$. The restriction and transfer in the L-Mackey functor $\underline{M}$, between the groups $\underline{M}(L)$ and $\underline{M}(e)$, are both L-equivariant (as discussed in the "Restriction, transfers, and Weyls" video), where again we think of L as acting trivially on $\underline{M}(L)$. Then the restriction and transfer between L and e in $\uparrow_L^K \underline{M}$ can be described as applying $\mathbb{Z}[K] \otimes_{\mathbb{Z}[L]} (-)$ to the restriction and transfer in $\underline{M}$.

Example 1.2.24. Consider the induction functor $\uparrow_{C_2}^{D_3}$. Thus let $\underline{M} \in \mathbf{Mack}(C_2)$. Then the induced D_3-Mackey functor $\uparrow_{C_2}^{D_3} \underline{M}$ is

$$\begin{tikzcd}
 & \underline{M}(C_2) \arrow[dl, bend right] \arrow[dr, bend left, "{(\mathrm{id}, r)}"'] & \\
\underline{M}(e) \arrow[ur, bend right] \arrow[dr, bend left, "\Delta"'] \arrow[loop left] & & \underline{M}(C_2) \oplus \underline{M}(e) \arrow[ul, bend right, "{(\mathrm{id}, t)}"'] \arrow[dl, bend right] \\
 & \mathbb{Z}[D_3] \otimes_{\mathbb{Z}[C_2]} \underline{M}(e) \arrow[ul, bend left, "\nabla"'] \arrow[ur, bend right] \arrow[loop below] &
\end{tikzcd}$$

To describe the restriction and transfer between the trivial subgroup and C_2, it is convenient to rewrite the value $\uparrow_{C_2}^{D_3} \underline{M}(e)$ at the trivial group as $\underline{M}(e) \oplus \underline{M}(e)^{\oplus 2}$, where the first copy corresponds to the identity coset of C_2 in D_3, and the other two copies are the non-identity cosets. Then the restriction and transfer are given as

$$\begin{tikzcd} \underline{M}(e) \oplus \underline{M}(e)^{\oplus 2} \arrow[rr, bend right, "t \oplus \nabla"'] & & \underline{M}(C_2) \oplus \underline{M}(e) \arrow[ll, bend right, "r \oplus \Delta"'] \end{tikzcd}$$

Thinking back to induction and restriction for representations from earlier in the course, recall that the results of Section 1.1.5 and Section 1.1.7 gave us that restriction is *both* left *and* right adjoint to induction of representations. The same is true for Mackey functors.

Proposition 1.2.25. *Let $H \leq G$. The restriction functor*

$$\downarrow_H^G(-) \colon \mathbf{Mack}(G) \longrightarrow \mathbf{Mack}(H)$$

is both left and right adjoint to the induction functor

$$\uparrow_H^G(-) \colon \mathbf{Mack}(H) \longrightarrow \mathbf{Mack}(G).$$

Now that we know that induction is left adjoint to restriction, we can find more projective Mackey functors.

Proposition 1.2.26. *For each $H \leq G$, the evaulation functor $\mathbf{ev}_H$ is right adjoint to tensoring with $\uparrow_H^G \underline{A}_H$:*

$$\begin{tikzcd} \mathbf{Mod}_{\mathbb{Z}} \arrow[r, shift left, "(-)\otimes \underline{A}_H"] \arrow[rr, bend left, "(-)\otimes\uparrow_H^G \underline{A}_H"] & \mathbf{Mack}(H) \arrow[l, shift left, "\mathbf{ev}_H"] \arrow[r, shift left, "\uparrow_H^G"] & \mathbf{Mack}(G) \arrow[l, shift left, "\downarrow_H^G"] \arrow[ll, bend left, "\mathbf{ev}_H"] \end{tikzcd}$$

Generalizing Proposition 1.2.17, we have

Proposition 1.2.27. *For any subgroup $H \leq G$, the G-Mackey functor $\uparrow_H^G \underline{A}_H$ is a projective Mackey functor.*

Corollary 1.2.28. *The abelian category $\mathbf{Mack}(G)$ has "enough" projectives. This means that for any $\underline{M} \in \mathbf{Mack}(G)$, there exists a projective Mackey functor $\underline{P} \in \mathbf{Mack}(G)$ and an epimorphism (levelwise-surjection) $\underline{P} \twoheadrightarrow \underline{M}$.*

Proof. The idea is to choose, for each $H \leq G$, a projective Mackey functor $\underline{P}_H$ and a map $\underline{P}_H \longrightarrow \underline{M}$ which is surjective at the subgroup H. For the Mackey functor $\underline{P}_H$, we take $\underline{M}(H) \otimes \uparrow_H^G \underline{A}_H$. Since tensoring with $\uparrow_H^G \underline{A}_H$ is left adjoint to evaluation at H, a map of Mackey functors

$$\underline{P}_H = \underline{M}(H) \otimes \uparrow_H^G \underline{A}_H \longrightarrow \underline{M}$$

corresponds to a group homomorphism

$$\underline{M}(H) \longrightarrow \mathbf{ev}_H \underline{M} = \underline{M}(H).$$

We choose the identity map of $\underline{M}(H)$, and then the adjoint map is surjective at level H. ■

The main reason why we care about having "enough" projectives is that it allows you to build a projective resolution of any object.

Example 1.2.29. Consider $\underline{\mathbb{Z}} \in \mathbf{Mack}(C_2)$. We have a surjection $\underline{A}_{C_2} \twoheadrightarrow \underline{\mathbb{Z}}$:

$$\begin{tikzcd} A(C_2) \cong \mathbb{Z}\{1, C_2\} \arrow[rr, "(1\ \ 2)"] \arrow[d, bend right, "(1\ \ 2)"'] & & \mathbb{Z} \arrow[d, bend right, "1"'] \\ \mathbb{Z} \arrow[rr, "1"] \arrow[u, bend right, "\binom{0}{1}"'] & & \mathbb{Z}, \arrow[u, bend right, "2"'] \end{tikzcd}$$

and this can be extended to a periodic projective resolution

$$\cdots \uparrow_e^{C_2}\mathbb{Z} \xrightarrow{1} \underline{A}_{C_2} \xrightarrow{C_2-2} \underline{A}_{C_2} \xrightarrow{1} \uparrow_e^{C_2}\mathbb{Z} \xrightarrow{1-\tau} \uparrow_e^{C_2}\mathbb{Z} \xrightarrow{1} \underline{A}_{C_2} \xrightarrow{C_2-2} \underline{A}_{C_2} \xrightarrow{1} \underline{\mathbb{Z}},$$

as we explain in a video.

Watch the video (in canvas): A projective resolution of $\underline{\mathbb{Z}}$ in $\mathbf{Mack}(C_2)$

Mon, Sept. 21

1.2.2. *Inflation.* There is another construction for Mackey functors, generalizing the pullback of representations (Definition 1.1.27), in the case of a quotient homomorphism $q\colon G \twoheadrightarrow Q = G/N$. Here, we will pullback a Q-Mackey functor to a G-Mackey functor. The basic idea is that one of the **Isomorphism Theorems** in group theory tells us that, given a normal subgroup $N \trianglelefteq G$, we have a bijective correspondence

$$\left\{ \begin{array}{c} \text{subgroups} \\ N \leq H \leq G \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{subgroups} \\ e \leq \overline{H} = H/N \leq G/N \end{array} \right\}$$

Thus a Mackey functor for $Q = G/N$ will look like the part of a Mackey functor for G which "lies above" N. We can visualize the subgroup lattice of G/N as being a "collapsed" or "deflated" version of the one for G. Thus to extend a Q-Mackey functor to a G-Mackey functor, we must "inflate" in the direction that was collapsed.

Definition 1.2.30. Let $N \trianglelefteq G$ and write $Q = G/N$ and $q\colon G \twoheadrightarrow Q$ for the quotient map. For any $\underline{M} \in \mathbf{Mack}(Q)$, we define $\mathrm{Inf}_Q^G \underline{M}$, the **inflation of $\underline{M}$ along** q, by the formula

$$\mathrm{Inf}_Q^G \underline{M}(H) = \begin{cases} \underline{M}(H/N) & N \leq H \\ 0 & \text{else.} \end{cases}$$

Notice that the Weyl group at $\underline{M}(e)$ is $Q = G/N$, and this is the same as the Weyl group at $\mathrm{Inf}_Q^G \underline{M}(N)$.

Example 1.2.31. Consider $G = C_2$ (or C_p for any prime), and take $N = G$. Then we can inflate a Mackey functor for G/G, in other words an abelian group, into a Mackey functor for G. This just sticks that abelian group at the top of the inflated Mackey functor, and a zero at the bottom.

$$\mathrm{Inf}_e^{C_2} A = \begin{array}{c} A \\ \\ 0 \end{array}$$

The same works for any group, really. For any coefficient group A and group G, we have a fully inflated Mackey functor $\mathrm{Inf}_{G/G}^G A$, which has value A at the top and zero at all proper subgroups.

Example 1.2.32. Consider $G = D_3$, and take $N = C_3$. Then we can inflate a Mackey functor $\underline{M}$ for $D_3/C_3 \cong C_2$ into a Mackey functor for D_3.

$$\mathrm{Inf}_{C_2}^{D_3} \underline{M} = \begin{tikzcd} & \underline{M}(C_2) \arrow[d, bend right] & \\ \underline{M}(e) \arrow[loop left] \arrow[u, bend right] & & 0 \\ & 0 & \end{tikzcd}$$

Mackey functors of the form $\mathrm{Inf}_{G/G}^G A$, which are concentrated at G/G, are known as "geometric" Mackey functors (at least in the homotopy theory community). The terminology comes from equivariant spectra and the notion of "geometric fixed points", which we will discuss later in the course.

Inflation functors have left and right adjoints [TW, p. 1871]. We describe these in the simple case of $\mathrm{Inf}_e^{C_2}$.

Proposition 1.2.33. *The inflation functor* $\mathrm{Inf}_e^{C_2}\colon \mathbf{AbGp} \longrightarrow \mathbf{Mack}(C_2)$ *has left and right adjoints. The left adjoint is*

$$\underline{M} \mapsto \underline{M}(C_2)/\operatorname{im} I_e^{C_2},$$

while the right adjoint is

$$\underline{M} \mapsto \ker R_e^{C_2}\colon \underline{M}(C_2) \longrightarrow \underline{M}(e).$$

1.3. **Group homology and cohomology.**

1.3.1. *Homology.* Our last algebra unit before we turn to topology will cover the homology and cohomology of groups. Recall that in Example 1.2.16, we saw that applying the quotient Mackey functor construction to the short exact

$$\mathbb{Z}_{\mathrm{sgn}} \hookrightarrow \mathbb{Z}[C_2] \twoheadrightarrow \mathbb{Z}$$

of $\mathbb{Z}[C_2]$-modules did not produce a short exact sequence of Mackey functors. In particular, restricting attention to the top values in the Mackey functors gave the sequence

$$\mathbb{F}_2 \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z},$$

which is not exact. Recall that the top value in the quotient Mackey functor (Definition 1.2.8) is the quotient of the $\mathbb{Z}[G]$-module by the G-action. This is also known as "coinvariants", which we describe now more precisely.

Definition 1.3.1. Given a $\mathbb{Z}[G]$-module M, the **coinvariants** of M is the abelian group

$$M/G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} M,$$

where the ring homomorphism $\mathbb{Z}[G] \longrightarrow \mathbb{Z}$ is the map sending each generator g to 1. Similarly, for a G-representation V, the coinvariants will be the vector space

$$V_G = \mathbf{k} \otimes_{\mathbf{k}[G]} V.$$

Explicitly, this is the quotient of V by the vector space spanned by elements of the form $\mathbf{v} - g \cdot \mathbf{v}$, for all $\mathbf{v} \in V$.

Then a way to summarize the non-exactness that we saw in Example 1.2.16 is to say that the functor $Q\colon \mathbf{Mod}_{\mathbb{Z}[G]} \longrightarrow \mathbf{Mack}(G)$ is not exact, or that the functor of coinvariants, $(-)/G\colon \mathbf{Mod}_{\mathbb{Z}[G]} \longrightarrow \mathbf{AbGp}$ is not exact. Since coinvariants can be described as a tensor product, as in Definition 1.3.1, this is a special case of the fact that, for any ring homomorphism $R \longrightarrow S$, the extension of scalars functor $S \otimes_R (-)\colon \mathbf{Mod}_R \longrightarrow \mathbf{Mod}_S$ is not exact (it is right exact but does not preserve injections).

When you have a right-exact functor, you can consider its "left-derived functors". In the case of a tensor product such as $S \otimes_R (-)$, these left-derived functors are the groups $\mathrm{Tor}_*^R(S, -)$.

Definition 1.3.2. Let G be a group. We define the **group homology of** G to be the groups

$$\mathrm{H}_n(G) = \mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}).$$

More generally, for any $\mathbb{Z}[G]$-module M, we define the homology of G with coefficients in M to be

$$\mathrm{H}_n(G; M) = \mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

These are the "higher derived functors of coinvariants." In particular, we have that

$$\mathrm{H}_0(G; M) \cong M/G. \tag{1.3.3}$$

Next time, we will compute some examples.

Wed, Sept. 23

Last time, we introduced group homology as Tor groups, namely

$$\mathrm{H}_n(G;M) = \mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

In general, to compute the groups $\mathrm{Tor}_n^R(L, M)$ for R-modules L or M, you

(1) Choose a projective resolution $P_* \longrightarrow L$ of L *or* a projective resolution $Q_* \longrightarrow M$ of M.
(2) Tensor your resolution with the module that you did *not* resolve. In other words, we are now looking at either the chain complex $P_* \otimes M$ or $L \otimes Q_*$
(3) Compute the homology of this chain complex. The homology groups of this complex are the desired Tor groups.

As is true for any ring R, free $\mathbb{Z}[G]$-modules are examples of projective modules.

Example 1.3.4. Take $G = C_2$, and we will switch to writing $\{e, g\}$ for the elements of C_2. We start by finding a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[C_2]$-module. We have a surjection

$$\varepsilon\colon \mathbb{Z}[C_2] \twoheadrightarrow \mathbb{Z}$$

sending 1 to 1. Note that since $\mathbb{Z}$ is a trivial C_2-module, equivariance of this map forces g to also go to 1. In other words, the kernel is $\mathbb{Z}\{e-g\}$. As a $\mathbb{Z}[C_2]$-module, this is the sign representation, since

$$g \cdot (e-g) = g - g^2 = g - e = -(e-g).$$

Now we can find a surjection

$$\delta\colon \mathbb{Z}[C_2] \twoheadrightarrow \mathbb{Z}\{e-g\}$$

by sending 1 to $e-g$. Then equivariance forces g to go to $-(e-g)$, so that the kernel of this surjection is $\mathbb{Z}\{e+g\}$. As a $\mathbb{Z}[C_2]$-module, this is the trivial representation $\mathbb{Z}$. Thus we may repeat the pattern to build a resolution. The resulting picture is

$$\begin{tikzcd}
 & P_2 \arrow[d, equal] & & P_1 \arrow[d, equal] & & P_0 \arrow[d, equal] & \\
\cdots \arrow[r, "e-g"] & \mathbb{Z}[C_2] \arrow[rr, "e+g"] \arrow[dr, "\delta"'] & & \mathbb{Z}[C_2] \arrow[rr, "e-g"] \arrow[dr, "\delta"'] & & \mathbb{Z}[C_2] \arrow[r, "\varepsilon"] & \mathbb{Z} \\
 & & \mathbb{Z}\{e+g\} \arrow[ur, hook] & & \mathbb{Z}\{e-g\} \arrow[ur, hook] & &
\end{tikzcd}$$

Now to compute homology, we tensor this resolution, over $\mathbb{Z}[C_2]$, with $\mathbb{Z}$. Recall that, by Definition 1.3.1, this is the same as passing to orbits or coinvariants. The resulting chain complex is

$$\begin{tikzcd}
 & P_4 \otimes_{\mathbb{Z}[C_2]} \mathbb{Z} \arrow[d, equal] & P_3 \otimes_{\mathbb{Z}[C_2]} \mathbb{Z} \arrow[d, equal] & P_2 \otimes_{\mathbb{Z}[C_2]} \mathbb{Z} \arrow[d, equal] & P_1 \otimes_{\mathbb{Z}[C_2]} \mathbb{Z} \arrow[d, equal] & P_0 \otimes_{\mathbb{Z}[C_2]} \mathbb{Z} \arrow[d, equal] \\
\cdots \arrow[r, "0"] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z}.
\end{tikzcd}$$

Thus we read off that

$$\mathrm{H}_n(C_2) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2 & n \text{ odd} \\ 0 & n \text{ even}, n > 0. \end{cases}$$

Example 1.3.5. Again take $G = C_2$, but consider $M = \mathbb{Z}_{\mathrm{sgn}}$. We use the same projective resolution of $\mathbb{Z}$, but now we tensor it with the C_2-module $\mathbb{Z}_{\mathrm{sgn}}$. The resulting chain complex is

$$\begin{array}{ccccccccc} & & P_3 \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}_{\mathrm{sgn}} & & P_2 \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}_{\mathrm{sgn}} & & P_1 \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}_{\mathrm{sgn}} & & P_0 \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}_{\mathrm{sgn}} \\ & & \| & & \| & & \| & & \| \\ \cdots & \xrightarrow{0} & \mathbb{Z}_{\mathrm{sgn}} & \xrightarrow{2} & \mathbb{Z}_{\mathrm{sgn}} & \xrightarrow{0} & \mathbb{Z}_{\mathrm{sgn}} & \xrightarrow{2} & \mathbb{Z}_{\mathrm{sgn}}. \end{array}$$

Thus we read off that

$$\mathrm{H}_n(C_2; \mathbb{Z}_{\mathrm{sgn}}) = \begin{cases} \mathbb{Z}/2 & n \text{ even}, \\ 0 & n \text{ odd}. \end{cases}$$

Example 1.3.6. Once again, consider $G = C_2$, but now take $M = \mathbb{R}$, with trivial action. The resulting chain complex is

$$\begin{array}{ccccccccccc} & & P_4 \otimes_{\mathbb{Z}[C_2]} \mathbb{R} & & P_3 \otimes_{\mathbb{Z}[C_2]} \mathbb{R} & & P_2 \otimes_{\mathbb{Z}[C_2]} \mathbb{R} & & P_1 \otimes_{\mathbb{Z}[C_2]} \mathbb{R} & & P_0 \otimes_{\mathbb{Z}[C_2]} \mathbb{R} \\ & & \| & & \| & & \| & & \| & & \| \\ \cdots & \xrightarrow{0} & \mathbb{R} & \xrightarrow{2} & \mathbb{R} & \xrightarrow{0} & \mathbb{R} & \xrightarrow{2} & \mathbb{R} & \xrightarrow{0} & \mathbb{R}. \end{array}$$

Thus we read off that

$$\mathrm{H}_n(C_2; \mathbb{R}) = \begin{cases} \mathbb{R} & n = 0 \\ 0 & n > 0. \end{cases}$$

There is another way to see this. Recall that we have a splitting $\mathbb{R}[C_2] \cong \mathbb{R} \oplus \mathbb{R}_{\mathrm{sgn}}$. In other words, $\mathbb{R}$ is a direct summand of the free module $\mathbb{R}[C_2]$. This implies that $\mathbb{R}$ is a projective $\mathbb{R}[C_2]$-module, so that $\mathrm{Tor}_{>0}^{\mathbb{R}[C_2]}(\mathbb{R}, \mathbb{R}) = 0$. Now extension of scalers from $\mathbb{Z}$ to $\mathbb{R}$ will convert a projective resolution of the $\mathbb{Z}[C_2]$-module $\mathbb{Z}$ to a projective resolution of the $\mathbb{R}[C_2]$-module $\mathbb{R}$. Since

$$P \otimes_{\mathbb{Z}[C_2]} \mathbb{R} \cong (P \otimes_{\mathbb{Z}} \mathbb{R}) \otimes_{\mathbb{R}[C_2]} \mathbb{R},$$

we get

$$\mathrm{Tor}_n^{\mathbb{Z}[C_2]}(\mathbb{Z}, \mathbb{R}) \cong \mathrm{Tor}_n^{\mathbb{R}[C_2]}(\mathbb{R}, \mathbb{R}).$$

So the fact that $\mathbb{R}$ is a projective $\mathbb{R}[C_2]$-module tells us that the higher homology groups $\mathrm{H}_{>0}(C_2; \mathbb{R})$ vanish. We similarly conclude that the higher homology groups $\mathrm{H}_{>0}(C_2; \mathbb{R}_{\mathrm{sgn}})$ vanish.

Example 1.3.7. Now consider $G = C_3$, with generator again called g. We start by finding a projective resolution of $\mathbb{Z}$. Here, if we start with the surjection $\varepsilon\colon \mathbb{Z}[C_3] \twoheadrightarrow \mathbb{Z}$ sending 1 to 1, then the kernel is $\mathbb{Z}\{e - g, e - g^2\}$. If we next take the surjection $\delta\colon \mathbb{Z}[C_3] \twoheadrightarrow \{e - g, e - g^2\}$ sending 1 to $e - g$, some linear algebra shows that the kernel is $\mathbb{Z}\{e + g + g^2\}$. As a $\mathbb{Z}[C_3]$-module, this is trivial. The resulting picture is

$$\begin{tikzcd} & P_2 \arrow[d, equal] & & P_1 \arrow[d, equal] & & P_0 \arrow[d, equal] & \\ \cdots \arrow[r, "e-g"] & \mathbb{Z}[C_3] \arrow[rr, "e+g+g^2"] \arrow[dr, "\delta"'] & & \mathbb{Z}[C_3] \arrow[rr, "e-g"] \arrow[dr, "\delta"'] & & \mathbb{Z}[C_3] \arrow[r, "\varepsilon"] & \mathbb{Z} \\ & & \mathbb{Z}\{e+g+g^2\} \arrow[ur, hook] & & \mathbb{Z}\{e-g, e-g^2\} \arrow[ur, hook] & & \end{tikzcd}$$

Tensoring with $\mathbb{Z}$ gives

$$\begin{array}{ccccccccccc} & & P_4 \otimes_{\mathbb{Z}[C_3]} \mathbb{Z} & & P_3 \otimes_{\mathbb{Z}[C_3]} \mathbb{Z} & & P_2 \otimes_{\mathbb{Z}[C_3]} \mathbb{Z} & & P_1 \otimes_{\mathbb{Z}[C_3]} \mathbb{Z} & & P_0 \otimes_{\mathbb{Z}[C_3]} \mathbb{Z} \\ & & \| & & \| & & \| & & \| & & \| \\ \cdots & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{3} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{3} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z}. \end{array}$$

Thus we read off that

$$\mathrm{H}_n(C_3) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/3 & n \text{ odd} \\ 0 & n \text{ even}, n > 0. \end{cases}$$

Now the methodology described in Example 1.3.4 and Example 1.3.7 generalizes to show that, for arbitrary $k > 0$,

$$\mathrm{H}_n(C_k) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/k & n \text{ odd} \\ 0 & n \text{ even}, n > 0. \end{cases}$$

Mon, Sept. 28

All of the examples thus far have been cyclic groups. Now we'll look at some non-cyclic groups.

Example 1.3.8. Take $G = K_4 = C_2 \times C_2$. Again the first step is to build a projective (or free) resolution of $\mathbb{Z}$ as a $\mathbb{Z}[K_4]$-module. We will write $K_4 = \{e, \ell, r, d\}$, with $d = \ell \cdot r$. The elements ℓ, d, and r generate the cyclic subgroups L, D, and R, respectively. The start of a resolution is given by

$$\begin{tikzcd}
 & P_2 \arrow[d, phantom, "\|"] & & P_1 \arrow[d, phantom, "\|"] & & P_0 \arrow[d, phantom, "\|"] & \\
\cdots \arrow[r] & \mathbb{Z}[K_4]\{x_2, w, y_2\} \arrow[rr, "\begin{pmatrix} e+\ell & -e-r & 0 \\ 0 & e+\ell & e+r \end{pmatrix}"] \arrow[dr] & & \mathbb{Z}[K_4]\{x, y\} \arrow[rr, "\begin{pmatrix} e-\ell & e-r \end{pmatrix}"] \arrow[dr, two heads, "\begin{pmatrix} e-\ell & e-r \end{pmatrix}"'] & & \mathbb{Z}[K_4] \arrow[r, "\varepsilon"] & \mathbb{Z} \\
 & & \ker \cong \mathbb{Z}^5 \arrow[ur, hook] & & \mathbb{Z}\{e-\ell, e-r, e-d\} \arrow[ur, hook] & &
\end{tikzcd}$$

Recall that all displayed maps are maps of $\mathbb{Z}[K_4]$-modules, so that each label for a map is telling you where the $\mathbb{Z}[K_4]$-module generators are sent. For example, in the map

$$P_1 = \mathbb{Z}[K_4]\{x, y\} \xrightarrow{(e-\ell,\ e-r)} I(K_4) = \mathbb{Z}\{e-\ell, e-r, e-d\},$$

the element $r \cdot x$ is sent to

$$r \cdot (e - \ell) = r - r\ell = r - d = (e-d) - (e-r).$$

If you want to see where this resolution comes from,

Watch the video (in canvas): The K_4-resolution of $\mathbb{Z}$

Since P_2 is a rank $3 \cdot 4 = 12$ free abelian group and it surjects onto the rank 5 kernel of the map out of P_1, it follows that the kernel of the map out of P_2 is free abelian of rank $12 - 5 = 7$. So the resolution does not appear to be periodic, and seems to be growing in size. Nevertheless, the portion of the resolution we have displayed is enough to calculate the first two homology groups. Tensoring our resolution, over $\mathbb{Z}[K_4]$, with $\mathbb{Z}$ gives the complex

$$\begin{tikzcd}
 & P_2 \otimes_{\mathbb{Z}[K_4]} \mathbb{Z} \arrow[d, phantom, "\|"] & P_1 \otimes_{\mathbb{Z}[K_4]} \mathbb{Z} \arrow[d, phantom, "\|"] & P_0 \otimes_{\mathbb{Z}[K_4]} \mathbb{Z} \arrow[d, phantom, "\|"] \\
\cdots \arrow[r] & \mathbb{Z}^3 \arrow[r, "\begin{pmatrix} 2 & -2 & 0 \\ 0 & 2 & 2 \end{pmatrix}"] & \mathbb{Z}^2 \arrow[r, "\begin{pmatrix} 0 & 0 \end{pmatrix}"] & \mathbb{Z}.
\end{tikzcd}$$

Since the map $P_1 \otimes_{\mathbb{Z}[K_4]} \mathbb{Z} \longrightarrow P_0 \otimes_{\mathbb{Z}[K_4]} \mathbb{Z}$ is the zero map, we conclude that

$$\mathrm{H}_0(K_4) \cong \mathbb{Z}.$$

The first homology group $\mathrm{H}_1(K_4)$ will be given by the cokernel of the displayed matrix. The image of this homomorphism (also known as the column space of the matrix) is the subgroup $2\mathbb{Z} \oplus 2\mathbb{Z} \subset \mathbb{Z} \oplus \mathbb{Z}$. We conclude that

$$\mathrm{H}_1(K_4) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Example 1.3.9. Take $G = D_3$. Once again, we will write down the start of a resolution of $\mathbb{Z}$, this time as a $\mathbb{Z}[D_3]$-module. Recall that we write r and s for the generators, where r has order 3 and s has order two. The start of a resolution is given by

$$\begin{tikzcd}
 & P_2 \arrow[d, phantom, "\|"] & & P_1 \arrow[d, phantom, "\|"] & & P_0 \arrow[d, phantom, "\|"] & \\
\cdots \arrow[r] & \mathbb{Z}[D_3]\{a, b, c, d\} \arrow[rr, "A"] \arrow[dr] & & \mathbb{Z}[D_3]\{x, y\} \arrow[rr, "\begin{pmatrix} e-r & e-s \end{pmatrix}"] \arrow[dr, two heads, "\begin{pmatrix} e-r & e-s \end{pmatrix}"'] & & \mathbb{Z}[D_3] \arrow[r, "\varepsilon"] & \mathbb{Z}, \\
 & & \ker \cong \mathbb{Z}^7 \arrow[ur, hook] & & I(D_3) \arrow[ur, hook] & &
\end{tikzcd}$$

where A is the matrix

$$A = \begin{pmatrix} e+r+r^2 & 0 & e-s-sr & e+r-s \\ 0 & e+s & r-e & r^2-e \end{pmatrix}.$$

[To find the matrix A, I first used linear algebra to find $\mathbb{Z}$-module generators for that rank 7 kernel. I then found that I only needed to use 4 of those elements to generate the kernel as a $\mathbb{Z}[D_3]$-module.] Tensoring our resolution, over $\mathbb{Z}[D_3]$, with $\mathbb{Z}$ gives the complex

$$\begin{tikzcd} & P_2 \otimes_{\mathbb{Z}[D_3]} \mathbb{Z} \arrow[d, equal] & P_1 \otimes_{\mathbb{Z}[D_3]} \mathbb{Z} \arrow[d, equal] & P_0 \otimes_{\mathbb{Z}[D_3]} \mathbb{Z} \arrow[d, equal] \\ \cdots \arrow[r] & \mathbb{Z}^4 \arrow[r, "\begin{pmatrix} 3 & 0 & -1 & 1 \\ 0 & 2 & 0 & 0 \end{pmatrix}"] & \mathbb{Z}^2 \arrow[r, "\begin{pmatrix} 0 & 0 \end{pmatrix}"] & \mathbb{Z}. \end{tikzcd}$$

Again the map $P_1 \otimes_{\mathbb{Z}[K_4]} \mathbb{Z} \longrightarrow P_0 \otimes_{\mathbb{Z}[K_4]} \mathbb{Z}$ is the zero map, and we conclude that

$$\mathrm{H}_0(D_3) \cong \mathbb{Z}.$$

The homology group $\mathrm{H}_1(D_3)$ is the cokernel of the map given by the matrix. The column space of this matrix is $\mathbb{Z} \oplus 2\mathbb{Z} \subset \mathbb{Z}^2$, and we conclude that

$$\mathrm{H}_1(D_3) \cong \mathbb{Z}/2\mathbb{Z}.$$

We can generalize from the examples we have seen.

Proposition 1.3.10. *For any group G, we have*

$$\mathrm{H}_0(G) \cong \mathbb{Z}.$$

Proof. This follows directly from the definition:

$$\mathrm{H}_0(G) = \mathrm{Tor}_0^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z} \otimes_{\mathbb{Z}[G]} \mathbb{Z} \cong \mathbb{Z}.$$

Alternatively, it can be seen from the resolution. Since $I(G) \subset \mathbb{Z}[G]$ is precisely the kernel of $\varepsilon \colon \mathbb{Z}[G] \longrightarrow \mathbb{Z}$, it follows that $I(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z} = 0$. It follows that $P_1 \otimes_{\mathbb{Z}[G]} \mathbb{Z} \longrightarrow P_0 \otimes_{\mathbb{Z}[G]} \mathbb{Z}$, which factors through $I(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$, must be the zero map. ■

The more substantive result is the following:

Proposition 1.3.11. *For any group G, there is a (natural) isomorphism*

$$\mathrm{H}_1(G) \cong I(G)/I(G)^2 \cong G_{ab}.$$

Proof. One of the properties of Tor is that a short exact sequence of R-modules $M \hookrightarrow N \twoheadrightarrow P$ induces a long exact sequence in Tor groups:

$$\mathrm{Tor}_n^R(A, M) \longrightarrow \mathrm{Tor}_n^R(A, N) \longrightarrow \mathrm{Tor}_n^R(A, P) \longrightarrow \mathrm{Tor}_{n-1}^R(A, M) \longrightarrow .$$

Applying this to the short exact sequence $I(G) \hookrightarrow \mathbb{Z}[G] \twoheadrightarrow \mathbb{Z}$ gives

$$\mathrm{H}_1(G; \mathbb{Z}[G]) \longrightarrow \mathrm{H}_1(G) \longrightarrow I(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z} \longrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} \mathbb{Z} \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}[G]} \mathbb{Z}.$$

Now the left homology group vanishes since $\mathbb{Z}[G]$ is a projective $\mathbb{Z}[G]$-module, and the rightmost map is an isomorphism. It follows that

$$\mathrm{H}_1(G) \cong I(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}.$$

Now for any $\mathbb{Z}[G]$-module, we have $M \otimes_{\mathbb{Z}[G]} \mathbb{Z} = M/M \cdot I(G)$. Thus we conclude that

$$\mathrm{H}_1(G) \cong I(G)/I(G)^2.$$

On the worksheet for this week, I will have you identify $I(G)/I(G)^2$ with the abelianization of G. ■

Wed, Sept. 30

Comparing Example 1.3.4 and Example 1.3.6, we saw that changing the coefficients from $\mathbb{Z}$ to $\mathbb{R}$ wiped out all of the higher homology groups. The same would be true for $G = C_k$, as can be seen from the discussion in Example 1.3.7. This generalizes to any finite group.

Proposition 1.3.12. *Suppse that A is an abelian group, or more generally a G-module, in which multiplication by the order $|G|$ of the group G is an isomorpihsm. Then the higher homology groups $\mathrm{H}_n(G; A)$, in which $n > 0$, all vanish.*

Proof. Write $R = \mathbb{Z}[\frac{1}{|G|}]$. Then our hyphotheses imply that A is an $R[G]$-module.

Recall the "norm" element $N = \sum_{g \in G} g \in \mathbb{Z}[G]$, and let $e = \frac{N}{|G|}$. Then, since $N \cdot N = |G| \cdot N$, we conclude that e is idempotent in $R[G]$. Furthermore, since N commutes with elements of G, we find that e is a *central* idempotent in $R[G]$. Now any central idempotent produces a splitting

$$R[G] \cong e \cdot R[G] \oplus (1 - e) \cdot R[G].$$

In particular, since $e \cdot R[G]$ is a summand of a free module, it is projective. It follows that the higher Tor groups $\mathrm{Tor}_n^{R[G]}(e \cdot R[G], A)$ vanish.

On the other hand, G fixes the norm N, and we similarly conclude that $e \cdot R[G]$ is (isomorphic to) the trivial G-module R. Thus

$$\mathrm{Tor}_n^{R[G]}(e \cdot R[G], A) = \mathrm{Tor}_n^{R[G]}(R, A) \cong \mathrm{Tor}_n^{R[G]}(\mathbb{Z} \otimes_{\mathbb{Z}[G]} R[G], A).$$

Now we use two facts from algebra:

(1) $R = \mathbb{Z}[\frac{1}{|G|}]$ is a flat $\mathbb{Z}$-module, and hence $R[G]$ is a flat $\mathbb{Z}[G]$-module (see `https://en.wikipedia.org/wiki/Flat_module#Flat_ring_extensions`)

(2) Tor satisfies flat base change, which in this case gives an isomorphism

$$\mathrm{Tor}_n^{\mathbb{Z}[G]}(M, A) \cong \mathrm{Tor}_n^{R[G]}(M \otimes_{\mathbb{Z}[G]} R[G], A)$$

for any G-module M and $R[G]$-module A. See Lemma 1.3.13 below for the general case.

Stringing together these isomorphisms, we find that

$$\mathrm{H}_n(G; A) = \mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, A) \cong 0. \qquad \blacksquare$$

We include the following for the sake of completeness, but feel free to skip over this result.

Lemma 1.3.13. *Let $S \longrightarrow T$ be a flat ring extension, meaning that T is flat as an S-module. Then for any S-module M and T-module N, we have an isomorphism*

$$\mathrm{Tor}_n^S(M, N) \cong \mathrm{Tor}_n^R(M \otimes_S T, N).$$

Proof. If $P_* \longrightarrow M$ is a projective resolution of M over S, then $\mathrm{Tor}_*^S(M, N)$ can be computed as the homology of $P_* \otimes_S N$. Now each $P_n \otimes_S T$ is a projective T-module, since P_n was projective over S, and $P_* \otimes_S T$ is a resolution of $M \otimes_S T$ because T is flat over S. Thus $\mathrm{Tor}_*^T(M \otimes_S T, N)$ can be computed as the homology of $(P_* \otimes_S T) \otimes_T N$, which is isomorphic to $P_* \otimes_S N$. Thus both sets of Tor groups are computed by isomorphic complexes and are therefore isomorphic. $\blacksquare$

Corollary 1.3.14. *For any (finite) group G, the higher homology groups with coefficients in $\mathbb{Q}$ or $\mathbb{R}$ vanish.*

This is often summarized in the slogan that

$$\boxed{\text{finite groups have no (reduced) rational homology.}}$$

1.3.2. *Cohomology.* In Definition 1.3.2, we defined group homology as a Tor group. Tor and Ext go hand-in-hand, and we use Ext to define the cohomology groups.

Definition 1.3.15. Let G be a group. We define the **group cohomology of** G to be the groups

$$\mathrm{H}^n(G) = \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}).$$

More generally, for any $\mathbb{Z}[G]$-module M, we define the cohomology of G with coefficients in M to be

$$\mathrm{H}^n(G; M) = \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

Just as homology was described as the "higher derived functors of coinvariants," we have a similar description for cohomology. First, note that $\mathrm{H}^0(G; M) = \mathrm{Ext}^0_{\mathbb{Z}[G]}(\mathbb{Z}, M) \cong \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$. Now since G acts trivially on $\mathbb{Z}$, a G-module map $\mathbb{Z} \longrightarrow M$ necessarily lands in the fixed points M^G, and such a homomorphism is completely determined by its value at $1 \in \mathbb{Z}$. We conclude that

Proposition 1.3.16. *For any group G, we have*

$$\mathrm{H}^0(G; M) = \mathrm{Ext}^0_{\mathbb{Z}[G]}(\mathbb{Z}, M) \cong M^G.$$

Thus we may say that cohomology groups are the "higher derived functors of fixed points."

From a computational standpoint, we have already done much of the work when considering homology. The reason is that, in order to compute an Ext group, such as $\mathrm{Ext}^n_R(M, N)$, you can take a projective resolution of M over R, and then compute the cohomology of the complex $\mathrm{Hom}_R(P_*, N)$. In the case of group homology, we would need a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$, but we already found such resolutions in Section 1.3.1.

Example 1.3.17. Let $G = C_k$. Recall from Example 1.3.7 that we have a periodic resolution

$$\xrightarrow{e-g} \mathbb{Z}[C_k] \xrightarrow{N} \mathbb{Z}[C_k] \xrightarrow{e-g} \mathbb{Z}[C_k] \xrightarrow{N} \mathbb{Z}[C_k] \xrightarrow{e-g} \mathbb{Z}[C_k] \xrightarrow{\varepsilon} \mathbb{Z}$$

of $\mathbb{Z}$ over $\mathbb{Z}[C_k]$. When we Hom this resolution into $\mathbb{Z}$, we recall that $\mathrm{Hom}_{\mathbb{Z}[C_k]}(\mathbb{Z}[C_k], M) \cong M$, where the isomorphism takes a function f to its value $f(1)$. Then we get

$$\begin{array}{ccccccccccc}
 & & \mathrm{Hom}_{C_k}(P_4, \mathbb{Z}) & & \mathrm{Hom}_{C_k}(P_3, \mathbb{Z}) & & \mathrm{Hom}_{C_k}(P_2, \mathbb{Z}) & & \mathrm{Hom}_{C_k}(P_1, \mathbb{Z}) & & \mathrm{Hom}_{C_k}(P_0, \mathbb{Z}) \\
 & & \| & & \| & & \| & & \| & & \| \\
\cdots & \xleftarrow{\ 0\ } & \mathbb{Z} & \xleftarrow{\ k\ } & \mathbb{Z} & \xleftarrow{\ 0\ } & \mathbb{Z} & \xleftarrow{\ k\ } & \mathbb{Z} & \xleftarrow{\ 0\ } & \mathbb{Z}.
\end{array}$$

For example, given $f \in \mathrm{Hom}_{C_k}(P_0, \mathbb{Z})$, we get a new function $d^0(f) \in \mathrm{Hom}_{C_k}(P_1, \mathbb{Z})$ as the composition $P_1 \xrightarrow{e-g} P_0 \xrightarrow{f} \mathbb{Z}$. But since the target of g is the trivial G-module $\mathbb{Z}$, it is necessarily the case that $f(g) = f(e)$, so that $d^0(f) = 0$. Thus we read off that

$$\mathrm{H}^n(C_k) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/k & n \text{ even}, n > 0 \\ 0 & n \text{ odd}. \end{cases}$$

Mon, Oct. 5

1.3.3. *Functoriality in G and the cup product.* In topology, homology is a covariant functor, whereas cohomology is contravariant. The same is true in this context.

Proposition 1.3.18. *Fix an abelian group A and an integer $n \geq 0$. Then homology and cohomology define functors*

$$\mathrm{H}_n(-;A)\colon \mathbf{Gp} \longrightarrow \mathbf{AbGp},$$

$$\mathrm{H}^n(-;A)\colon \mathbf{Gp}^{op} \longrightarrow \mathbf{AbGp}.$$

Proof. We give the argument for homology, and a similar deduction applies to cohomology.

Let $\varphi\colon G \longrightarrow K$ be a group homomorphism. We wish to produce a homomorphism $\varphi_*\colon \mathrm{H}_n(G;A) \longrightarrow \mathrm{H}_n(K : A)$. Let $P_* \longrightarrow \mathbb{Z}$ be a resolution of $\mathbb{Z}$ over K. The pullback of modules along φ is exact, so $\varphi^* P_*$ is a resolution of $\mathbb{Z}$ over G. Furthermore, we have a chain map

$$\varphi^*(P_*) \otimes_{\mathbb{Z}[G]} A \cong \Big(\mathbb{Z}[K] \otimes_{\mathbb{Z}[G]} \varphi^*(P_*)\Big) \otimes_{\mathbb{Z}[K]} A \longrightarrow P_* \otimes_{\mathbb{Z}[K]} A.$$

The target of this map computes $\mathrm{H}_*(K;A)$. However, unfortunately there is no reason to expect the resolution $\varphi^*(P_*)$ of $\mathbb{Z}$ over G to be *projective*, and therefore the source of this chain map may not compute the homology groups of G. To amend this, we need to use the magic of projective resolutions. Namely, we have the following lifting lemma:

Lemma 1.3.19. *Let $P_* \longrightarrow M$ be a projective resolution, and let $R_* \longrightarrow M$ be any other resolution. Then there exists a map of resolutions $P_* \longrightarrow R_*$, and furthermore this map of resolutions is unique up to chain homotopy.*

Sketch. This is one of the fundamental results in homological algebra, so we just give a sketch. The first step is to build a map $f_0\colon P_0 \longrightarrow Q_0$ such that

$$\begin{tikzcd}
P_0 \arrow[rr, dashed, "f_0"] \arrow[dr, two heads, "\varepsilon_Q"] & & Q_0 \arrow[dl, two heads, "\varepsilon_Q"] \\
& M &
\end{tikzcd}$$

commutes. But since ε_Q is surjective, the defining property of projectives provides us such a map f_0.

Next, the composition $P_1 \longrightarrow P_0 \xrightarrow{f_0} Q_0$ factors through $\ker(\varepsilon_Q)$.

$$\begin{tikzcd}
P_1 \arrow[rr, dashed, "f_1"] \arrow[dr, dashed] \arrow[dd] & & Q_1 \arrow[dl, two heads] \arrow[dd] \\
& \ker(\varepsilon_Q) \arrow[dr, hook] & \\
P_0 \arrow[rr, "f_0"] \arrow[dr, two heads, "\varepsilon_Q"] & & Q_0 \arrow[dl, two heads, "\varepsilon_Q"] \\
& M &
\end{tikzcd}$$

Since Q_1 surjects onto $\ker(\varepsilon_Q)$, projectivity again gives the lift f_1. These f_* are by no means unique, but projectivity again allows you to build a chain homotopy between any two sets of choices. $\square$

Going back to our proof, let now $Q_* \longrightarrow \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ over G. By the Lemma, we have a chain map $f_*: Q_* \longrightarrow \varphi^*(P_*)$. Now the composition

$$Q_* \otimes_{\mathbb{Z}[G]} \mathbb{Z} \xrightarrow{f_* \otimes \mathrm{id}} \varphi^*(P_*) \otimes_{\mathbb{Z}[G]} \mathbb{Z} \longrightarrow P_* \otimes_{\mathbb{Z}[K]} \mathbb{Z}$$

induces the desired map on homology.

Of course, since we made choices, you may wonder whether this will really be functorial. But the chain homotopy part of the Magic Projectives Lemma saves the day. I will leave it to you to work out the details if you are interested. ■

One of the reasons to care about Proposition 1.3.18 is that it will give us what we are really after: the ring structure on cohomology. For this, we first need to define the external product, also known as the cross product.

Definition 1.3.20. Let G and K be groups, and let R be any commutative ring. Let P_* be a free resolution of $\mathbb{Z}$ over G, and let Q_* be a free resolution of $\mathbb{Z}$ over K. Then we observe that

(1) For each n and ℓ, the group $P_n \otimes Q_\ell$ is a free $\mathbb{Z}[G \times K] \cong \mathbb{Z}[G] \otimes \mathbb{Z}[K]$-module
(2) Let us write $(P \otimes Q)_*$ for the "total complex" associated to the double complex $P_* \otimes Q_*$, as in the "K_4-resolution of $\mathbb{Z}$" video. Then $(P \otimes Q)_*$ is a free resolution of $\mathbb{Z}$ over $G \times K$.
(3) We can tensor morphisms to get

$$\mathrm{Hom}_G(P_*, \mathbb{Z}) \otimes \mathrm{Hom}_K(Q_*, \mathbb{Z}) \longrightarrow \mathrm{Hom}_{G\times K}((P \otimes Q)_*, \mathbb{Z}).$$

(4) Using coefficients in the ring R instead, we can tensor and multiply to get

$$\mathrm{Hom}_G(P_*, R) \otimes \mathrm{Hom}_K(Q_*, R) \longrightarrow \mathrm{Hom}_{G\times K}((P \otimes Q)_*, R \otimes R) \longrightarrow \mathrm{Hom}_{G\times K}((P \otimes Q)_*, R).$$

We then define the **cross product** to be the composition

$$\begin{tikzcd}
\mathrm{H}^n(G;R) \otimes \mathrm{H}^\ell(K;R) \arrow[r] \arrow[ddr, bend right, "\times"'] & \mathrm{H}^{n+\ell}\left(\mathrm{Hom}_G(P_*, R) \otimes \mathrm{Hom}_K(Q_*, R)\right) \arrow[d] \\
& \mathrm{H}^{n+\ell}(\mathrm{Hom}_{G\times K}((P \otimes Q)_*, R)) \arrow[d, equal] \\
& \mathrm{H}^{n+\ell}(G \times K; R)
\end{tikzcd}$$

Like in topology, we can internalize the external product by pulling back along a diagonal.

Definition 1.3.21. Let G be a group and R a commutative ring. Denote by $\Delta: G \longrightarrow G \times G$ the diagonal homomorphism. We then define the **cup product** as the composition

$$\begin{tikzcd}
\mathrm{H}^n(G;R) \otimes \mathrm{H}^\ell(G;R) \arrow[r, "\times"] \arrow[dr, bend right, "\cup"'] & \mathrm{H}^{n+\ell}(G \times G; R) \arrow[d, "\Delta^*"] \\
& \mathrm{H}^{n+\ell}(G; R).
\end{tikzcd}$$

Proposition 1.3.22. *For any group G and commutative ring R of coefficients, the cup product makes $\mathrm{H}^*(G;R)$ into a graded-commutative ring. Furthermore, for any group homomorphism $\varphi: G \longrightarrow K$, the induced map $\varphi^*: \mathrm{H}^*(K;R) \longrightarrow \mathrm{H}^*(G;R)$ is a ring homomorphism.*

Recall that graded-commutative here means that if $x \in \mathrm{H}^n(G;R)$ and $y \in \mathrm{H}^\ell(G;R)$, then

$$y \cup x = (-1)^{\deg(y)\cdot\deg(x)} x \cup y.$$

Like in topology, we will rarely use the cup symbol, electing instead to write $x \cdot y$ for $x \cup y$.

Wed, Oct. 7

Last time, we gave one of the most important properties of cohomology: it gives a *ring* as output. Let's look at some examples.

Example 1.3.23. Of course, we start with $G = C_2$. As we saw in Example 1.3.17, the cohomology groups are $\mathbb{Z}$ in degree zero and $\mathbb{Z}/2$ in every positive even degree (and zero in odd degrees). The element $1 \in \mathrm{H}^0(C_2)$ is the unit, so multiplication of any cohomology class by any $n \in \mathrm{H}^0(C_2)$ is just multiplication by that integer. The only question is how the classes in positive degrees multiply. Let $x \in \mathrm{H}^2(C_2) \cong \mathbb{Z}/2$ be a generator (meaning the nonzero class). Then we wish to determine whether $x \cdot x \in \mathrm{H}^4(C_2)$ is zero or not. Writing P_* for the free resolution of $\mathbb{Z}$ over C_2 introduced in Example 1.3.4, the cohomology class x is represented by the C_2-equivariant homomorphism $f_x \colon P_2 = \mathbb{Z}[C_2] \longrightarrow \mathbb{Z}$ sending $1 \in \mathbb{Z}[C_2]$ to $1 \in \mathbb{Z}$. (We call this map ε when discussing resolutions.)

The description of the cup product from last class says that the cohomology class $x \cdot x \in \mathrm{H}^4(C_2)$ is given by

$$(P \otimes P)_4 \twoheadrightarrow P_2 \otimes P_2 \xrightarrow{f_x \otimes f_x} \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}.$$

Here $(P \otimes P)_*$ is a free resolution of $\mathbb{Z}$ over $C_2 \times C_2$, but when we restrict it to C_2 along the diagonal $C_2 \xrightarrow{\Delta} C_2 \times C_2$, it gives a free resolution of $\mathbb{Z}$ over C_2, since the free module $\mathbb{Z}[C_2 \times C_2]$ restricts to a rank two free module along the diagonal inclusion. However, the trouble is that it is not clear from this description if this particular degree 4 cohomology class is trivial or not.

There are two options. One is to write out the complex up to degree 4 and see that this degree 4 class is not in the image of the differential. Another is to compare this (large) resolution to our small resolution P_*. We choose the latter option. The resolution $(P \otimes P)_*$ is displayed on the right, and the dashed arrows are built, starting from the bottom, at each step choosing a map that makes the square below commute.

$$\begin{tikzcd}
P_4 = \mathbb{Z}[C_2] \arrow[r, dashed, "\begin{pmatrix} 1\otimes 1 & g\otimes g & -1\otimes 1 & -g\otimes g & 1\otimes 1 \end{pmatrix}"] \arrow[d, "1+g"] & (P_0 \otimes P_4) \oplus (P_1 \otimes P_3) \oplus (P_2 \otimes P_2) \oplus (P_3 \otimes P_1) \oplus (P_4 \otimes P_0) \arrow[d, "\begin{pmatrix} 1\otimes(1+g) & (1-g)\otimes 1 & 0 & 0 & 0 \\ 0 & -1\otimes(1-g) & (1+g)\otimes 1 & 0 & 0 \\ 0 & 0 & 1\otimes(1+g) & (1-g)\otimes 1 & 0 \\ 0 & 0 & 0 & -1\otimes(1-g) & (1+g)\otimes 1 \end{pmatrix}"] \\
P_3 = \mathbb{Z}[C_2] \arrow[r, dashed, "\begin{pmatrix} 1\otimes 1 & -g\otimes g & -1\otimes 1 & g\otimes g \end{pmatrix}"] \arrow[d, "1-g"] & (P_0 \otimes P_3) \oplus (P_1 \otimes P_2) \oplus (P_2 \otimes P_1) \oplus (P_3 \otimes P_0) \arrow[d, "\begin{pmatrix} 1\otimes(1-g) & (1-g)\otimes 1 & 0 & 0 \\ 0 & -1\otimes(1+g) & (1+g)\otimes 1 & 0 \\ 0 & 0 & 1\otimes(1-g) & (1-g)\otimes 1 \end{pmatrix}"] \\
P_2 = \mathbb{Z}[C_2] \arrow[r, dashed, "\begin{pmatrix} 1\otimes 1 & g\otimes g & -1\otimes 1 \end{pmatrix}"] \arrow[d, "1+g"] & (P_0 \otimes P_2) \oplus (P_1 \otimes P_1) \oplus (P_2 \otimes P_0) \arrow[d, "\begin{pmatrix} 1\otimes(1+g) & (1-g)\otimes 1 & 0 \\ 0 & -1\otimes(1-g) & (1+g)\otimes 1 \end{pmatrix}"] \\
P_1 = \mathbb{Z}[C_2] \arrow[r, dashed, "\begin{pmatrix} 1\otimes 1 & -g\otimes g \end{pmatrix}"] \arrow[d, "1-g"] & (P_0 \otimes P_1) \oplus (P_1 \otimes P_0) \arrow[d, "\begin{pmatrix} 1\otimes(1-g) & (1-g)\otimes 1 \end{pmatrix}"] \\
P_0 = \mathbb{Z}[C_2] \arrow[r, dashed, "1\otimes 1"] \arrow[dr] & \mathbb{Z}[C_2] \otimes \mathbb{Z}[C_2] = P_0 \otimes P_0 \arrow[d] \\
& \mathbb{Z}
\end{tikzcd}$$

Given this comparison of projective resolutions, the class $x \cdot x$ will be represented by

$$P_4 = \mathbb{Z}[C_2] \xrightarrow{-1\otimes 1} P_2 \otimes P_2 \xrightarrow{f_x \otimes f_x} \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}.$$

This composition sends 1 to -1, which is a generator of $\mathbb{Z}$. Thus we conclude that this degree four cohomology class is the (unique) nontrivial cohomology class in $\mathrm{H}^4(C_2)$. A similar analysis shows that the nonzero class in $\mathrm{H}^{2n}(C_2)$ is x^n. In other words, we conclude that

$$\boxed{\mathrm{H}^*(C_2) \cong \mathbb{Z}[x]/(2x)},$$

where x is in degree 2.

Example 1.3.24. We saw in Example 1.3.17 that the cohomology of any C_k is also concentrated in even degrees. The answer that we saw in Example 1.3.23 also holds for C_k, in the sense that

$$\mathrm{H}^*(C_k) \cong \mathbb{Z}[x]/(kx).$$

You should think of this as being essentially the ring $\mathbb{Z}/k[x]$, except that you have a $\mathbb{Z}$ in degree zero rather than the $\mathbb{Z}/k$.

Example 1.3.25. The answer becomes a little neater if we work with finite coefficients. For C_2, we choose the coefficients $\mathbb{F}_2$ (by the cohomological analogue of Proposition 1.3.12, if we choose a field of any other characteristic, the higher cohomology groups will vanish). We could easily calculate the cohomology groups directly using our favorite resolution, but since we have already done the work of calculating the integral cohomology groups, let's use that to deduce the mod 2 cohomology groups. The key is that the short exact sequence

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{F}_2$$

of coefficients induces a long exact sequence in cohomology. For $n > 0$ even, the long exact sequence gives

$$\mathrm{H}^n(C_2) \cong \mathbb{Z}/2 \xrightarrow{2=0} \mathrm{H}^n(C_2) \cong \mathbb{Z}/2 \longrightarrow \mathrm{H}^n(C_2;\mathbb{F}_2) \longrightarrow \mathrm{H}^{n+1}(C_2) = 0,$$

from which we conclude that $\mathrm{H}^n(C_2;\mathbb{F}_2) \cong \mathbb{Z}/2$. For n odd, the long exact sequence gives

$$\mathrm{H}^n(C_2) = 0 \longrightarrow \mathrm{H}^n(C_2;\mathbb{F}_2) \longrightarrow \mathrm{H}^{n+1}(C_2) \xrightarrow{2=0} \mathrm{H}^{n+1}(C_2),$$

from which we again conclude that $\mathrm{H}^n(C_2;\mathbb{F}_2) \cong \mathbb{Z}/2$. Now every cohomology group is $\mathbb{Z}/2$.

In fact, the cohomology ring is

$$\boxed{\mathrm{H}^*(C_2;\mathbb{F}_2) \cong \mathbb{F}_2[x_1]},$$

where the class x_1 is in degree 1. Since the ring map $\mathrm{H}^*(C_2) \longrightarrow \mathrm{H}^*(C_2;\mathbb{F}_2)$ is surjective in positive, even degrees, we see that every class in higher even degree is a power of the class in degree 2. Also, the connecting homomorphism is a $\mathrm{H}(C_2)$-module map, so writing y_3 for the class in degree 3, we have

$$\begin{tikzcd}
 & x_2^2 \\
y_3 \arrow[ur, mapsto, "\delta"] & \\
 & x_2 \arrow[uu, bend right, "\cdot x_2"'] \\
x_1 \arrow[uu, bend right, "\cdot x_2"'] \arrow[ur, mapsto, "\delta"] & \\
\mathrm{H}^*(C_2;\mathbb{F}_2) & \mathrm{H}^*(C_2),
\end{tikzcd}$$

from which we learn that y_3 is $x_2 \cdot x_1$. The same will hold for the classes in higher odd degrees. The only remaining issue is whether $x_1 \cdot x_1$ is zero or not. I will ask you to verify this on the worksheet this week.

$\boxed{\textbf{Mon, Oct. 12}}$

Last time, we discussed the cohomology rings $\mathrm{H}^*(C_2)$ and $\mathrm{H}^*(C_2;\mathbb{F}_2)$. On Friday's worksheet, Problem 3 (which has been moved to Homework 3) asked you to investigate the cohomology ring $\mathrm{H}^*(C_6) \cong \mathbb{Z}[x_2]/(6x_2)$, in the sense that you were asked to consider the homomorphisms

$$\mathrm{H}^*(C_2) \longrightarrow \mathrm{H}^*(C_6) \longrightarrow \mathrm{H}^*(C_3)$$

arising from the group extension $C_3 \hookrightarrow C_6 \twoheadrightarrow C_2$. In general, it is true that an extension $N \hookrightarrow G \twoheadrightarrow Q$ gives rise to homomorphisms

$$\mathrm{H}^*(Q) \longrightarrow \mathrm{H}^*(G) \longrightarrow \mathrm{H}^*(N),$$

though this is *not* a short exact sequence. For example, in degree zero both maps are the identity on $\mathbb{Z}$. In positive degrees, the two maps do compose to zero. This follows from functoriality. Since the composition $N \longrightarrow G \longrightarrow Q$ is the constant map to the identity, it follows that the composition of maps on cohomology factors through the cohomology of the trivial group. But this vanishes in positive degrees.

Example 1.3.26. Consider $G = C_4$ and the group extension

$$C_2 \overset{\iota}{\hookrightarrow} C_4 \overset{p}{\twoheadrightarrow} C_2.$$

We first determine $\iota^*\colon \mathrm{H}^*(C_4) \cong \mathbb{Z}[x_2]/(4x_2) \longrightarrow \mathrm{H}^*(C_2) \cong \mathbb{Z}[x_2]/(2x_2)$. To do this, we consider the comparison of resultions $P_*^{C_2} \longrightarrow \downarrow_{C_2}^{C_4} P_*^{C_4}$ displayed below in low degrees. In this diagram, we write g for the generator of C_4 and c for the generator of C_2, so that $\iota(c) = g^2$.

$$\begin{tikzcd}
\downarrow_{C_2}^{C_4} \mathbb{Z}[C_4] \arrow[r, "1+g+g^2+g^3"] \arrow[d, equal] & \downarrow_{C_2}^{C_4} \mathbb{Z}[C_4] \arrow[r, "1-g"] \arrow[d, equal] & \downarrow_{C_2}^{C_4} \mathbb{Z}[C_4] \arrow[d, equal] \arrow[dr, "\varepsilon"] & \\
\mathbb{Z}[C_2] \oplus g\mathbb{Z}[C_2] \arrow[r, "\begin{pmatrix} 1+c & 1+c \\ 1+c & 1+c \end{pmatrix}"] & \mathbb{Z}[C_2] \oplus g\mathbb{Z}[C_2] \arrow[r, "\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}"] & \mathbb{Z}[C_2] \oplus g\mathbb{Z}[C_2] & \mathbb{Z} \\
\mathbb{Z}[C_2] \arrow[r, "1+c"] \arrow[u, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & \mathbb{Z}[C_2] \arrow[r, "1-c"] \arrow[u, "\begin{pmatrix} 1 \\ 1 \end{pmatrix}"] & \mathbb{Z}[C_2] \arrow[u, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] \arrow[ur, "\varepsilon"] &
\end{tikzcd}$$

The element $x_2 \in \mathrm{H}^2(C_4)$ is represented by the map $\mathbb{Z}[C_4] \longrightarrow \mathbb{Z}$ sending 1 to 1. Precomposing with the left vertical map in the diagram gives the map $\mathbb{Z}[C_2] \longrightarrow \mathbb{Z}$ sending 1 to 1, which represents $x_2 \in \mathrm{H}^2(C_2)$. This shows that $\iota^*(x_2) = x_2$. Since x_2 is an algebra generator for $\mathrm{H}^*(C_4)$, this completely determines the ring homomorphism $\iota^*\colon \mathrm{H}^*(C_4) \longrightarrow \mathrm{H}^*(C_2)$.

We now determine the ring homomorphism $p^*\colon \mathrm{H}^*(C_2) \longrightarrow \mathrm{H}^*(C_4)$. For this we build a comparison of resolutions $P_*^{C_4} \longrightarrow p^* P_*^{C_2}$.

$$\begin{tikzcd}
p^*\mathbb{Z}[C_2] \arrow[r, "1+c"] \arrow[d, equal] & p^*\mathbb{Z}[C_2] \arrow[r, "1-c"] \arrow[d, equal] & p^*\mathbb{Z}[C_2] \arrow[d, equal] \arrow[dr, "\varepsilon"] & \\
\mathbb{Z}[C_4/C_2] \arrow[r, "1+g"] & \mathbb{Z}[C_4/C_2] \arrow[r, "1-g"] & \mathbb{Z}[C_4/C_2] & \mathbb{Z} \\
\mathbb{Z}[C_4] \arrow[r, "1+g+g^2+g^3"] \arrow[u, "2"] & \mathbb{Z}[C_4] \arrow[r, "1-g"] \arrow[u, two heads, "1"] & \mathbb{Z}[C_4] \arrow[u, two heads, "1"] \arrow[ur, "\varepsilon"] &
\end{tikzcd}$$

The element $x_2 \in \mathrm{H}^2(C_2)$ is represented by the map $\mathbb{Z}[C_2] \longrightarrow \mathbb{Z}$ sending 1 to 1. Precomposing with the left vertical map in the diagram gives the map $\mathbb{Z}[C_4] \longrightarrow \mathbb{Z}$ sending 1 to 2, which represents $2x_2 \in \mathrm{H}^2(C_4)$. This shows that $p^*(x_2) = 2x_2$. Note that since p^* is a ring map, this shows

that
$$p^*(x_2^2) = p^*(x_2)^2 = (2x_2)^2 = 4x_2^2 = 0.$$
In particular, p^* is an inclusion in degree 2 and the zero map in higher degrees.

Example 1.3.27. The other group of order 4, $G = K_4$, is the split extension of C_2 by C_2, meaning that K_4 is simply the product $K_4 \cong C_2 \times C_2$. Let p_1 and p_2 denote the projections onto the first and second factors. Pulling back along these projections induces ring maps $p_1^*, p_2^* \colon \mathrm{H}^*(C_2) \longrightarrow \mathrm{H}^*(K_4)$. Since the tensor product is the coproduct in the category of commutative ring, together these define a ring map
$$\mathrm{H}^*(C_2) \otimes \mathrm{H}^*(C_2) \xrightarrow{\mathrm{H}} {}^*(K_4).$$
The **Künneth theorem** (see the description on nLab) tells us that, at least if we work with field coefficients, this map is an isomorphism
$$\mathrm{H}^*(C_2; \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathrm{H}^*(C_2; \mathbb{F}_2) \xrightarrow{\cong} \mathrm{H}^*(K_4; \mathbb{F}_2).$$
In other words, we conclude that
$$\mathrm{H}^*(K_4; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, y_1],$$
where x_1 and y_1 are both in degree 1.

The following result will be used in the ensuing example. We delay its proof for now.

Proposition 1.3.28. *Let p be prime, and let $\iota \colon S \hookrightarrow G$ be the inclusion of a Sylow p-subgroup. Then the restriction*
$$\iota^* \colon \mathrm{H}^*(G; \mathbb{F}_p) \longrightarrow \mathrm{H}^*(S; \mathbb{F}_p)$$
is injective. Moreover, if S is normal in G, the image is the invariants $\mathrm{H}^*(S; \mathbb{F}_p)^W$ under the action of the Weyl group $W = W_G(S) = G/S$.

Example 1.3.29. Consider the case of $G = D_3$. There are two important differences in comparing this case to the previous examples. First, D_3 is not abelian, and, second, its order is not a power of a prime.

We have the extension
$$C_3 \hookrightarrow D_3 \overset{p}{\twoheadrightarrow} C_2,$$
and it is split, in the sense that p has a section given by the inclusion of any of the cyclic subgroups of order two (there are three of them). However, in the nonabelian setting, a splitting provides a set-level decomposition but not necessarily a group-level one. (It says that D_3 is a semi-direct product.)

Since 2 and 3 are the primes dividing $|D_3|$, the relevant computations are $\mathrm{H}^*(D_3; \mathbb{F}_2)$ and $\mathrm{H}^*(D_3; \mathbb{F}_3)$. For the 2-primary computation, the splitting shows that the homomorphism $s^* \colon \mathrm{H}^*(C_2; \mathbb{F}_2) \longrightarrow \mathrm{H}^*(D_3; \mathbb{F}_2)$ is surjective. On the other hand, Proposition 1.3.28 tells us that $s^* \colon \mathrm{H}^*(D_3; \mathbb{F}_2) \longrightarrow \mathrm{H}^*(C_2; \mathbb{F}_2)$ is also injective (actually an isomorphism, since the Weyl group of $C_2 \leq D_3$ is trivial). We conclude that $\mathrm{H}^*(D_3; \mathbb{F}_2) \cong \mathbb{F}_2[x_1]$.

For the 3-primary computation, again Proposition 1.3.28 tells us that the restriction $\iota^* \colon \mathrm{H}^*(D_3; \mathbb{F}_3) \longrightarrow \mathrm{H}^*(C_3; \mathbb{F}_3)$ is injective. However, in this case it is in fact not an isomorphisms, since the Weyl invariants will turn out to be a proper subring. Here $C_3 \leq D_3$ is normal, and the Weyl group is $W = D_3/C_3 \cong C_2$. The C_2-action on C_3 is the inverse map, sending a generator g to $g^{-1} = g^2$. Recalling that $\mathrm{H}^*(C_3; \mathbb{F}_3) \cong \mathbb{F}_3[x_1, y_2]/(x_1^2)$, it remains to determine the C_2-action on the generators x_1 and y_2.

Watch the video (in canvas): The Weyl action on the cohomology of C_3

According to the video, the generator of C_2 multiplies each of x_1 and y_2 by 2, so that neither is fixed by the action. On the other hand, x_1y_2 and y_2^2 are both fixed, and we see that

$$\mathrm{H}^*(D_3;\mathbb{F}_3) \cong \mathbb{F}_3[x_3, y_4]/(x_3^2),$$

where $\iota^*(x_3) = x_1y_2$ and $\iota^*(y_4) = y_2^2$.

This completes the algebra module. Next time, we turn to group actions in topology!

Wed, Oct. 14

2.1. **Group actions in topology: G-spaces.** Applying Definition 1.1.1 to the category **Top** of topological spaces gives rise to the notion of G-space. Alternatively, a G-space X consists of a space X and a continuous map $a\colon G \times X \longrightarrow X$, where we give G the discrete topology, which is appropriately unital and associative. Since G is discrete, the map a can also be described as a map $\coprod_G X \longrightarrow X$. For G-spaces X and Y, a G-equivariant map is simply a continuous map which is compatible with the G-action in the sense that $f(g \cdot x) = g \cdot f(x)$.

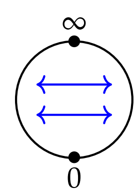
Definition 2.1.1. We will denote by G**Top** the category of G-spaces and G-equivariant maps.

Example 2.1.2. Of course, for any space X, you can always consider X as a G-space, where we give X the trivial action. For instance, we have S^n equipped with a trivial action.

Example 2.1.3. For any $n \geq 0$, we have the **antipodal action** of C_2 on S^n, where the nonidentity element acts as multiplication by -1. This is an example of a **free action**, in the sense that the nonidentity element does not fix any points.

Example 2.1.4. Let V be a representation of G. We denote by S^V the one-point compactification of V. If V is an n-dimensional (real) representation, then S^V is an n-dimensional sphere. We claim that the G-action on V extends to a G-action on S^V, where the new point ∞ is fixed by G. To see this recall from MA551[2] that a map $f\colon X \longrightarrow Y$ between locally compact, Hausdorff spaces extends to a map $\tilde{f}\colon \tilde{X} \longrightarrow \tilde{Y}$ between the one-point compactifications, such that $f(\infty_X) = \infty_Y$, if and only if f is a *proper* map, meaning that the preimage of any compact set is compact. Now for any $g \in G$, the action by g on V gives a homeomorphism $V \cong V$, which is certainly a proper map. It follows that the action by g extends to a homeomorphism $S^V \cong S^V$ which fixes ∞.

Example 2.1.5. Consider the sign representation σ of C_2. Then we have the sign represetntation sphere S^σ. Since zero is the only fixed point in the sign representation, the representation sphere S^σ has exactly two fixed points, the point zero and the new point at infinity. We can also think of this as the unit circle inside $\mathbb{C}$, where C_2 acts as complex conjugation. Then there are precisely two fixed points, 1 and -1.



Example 2.1.6. Let ρ denote the regular representation of C_2. Then S^ρ is a 2-dimensional sphere. It may be thought of as the one-point compactification of $\mathbb{C}$, also known as $\mathbb{CP}^1$, equipped with the complex conjugation action. Since $\mathbb{R} \subset \mathbb{C}$ is fixed by conjugation, we get that $\mathbb{RP}^1 \subset \mathbb{CP}^1$ is fixed. This is an S^1 inside the S^2.

Example 2.1.7. Let λ_3 be the irreducible 2-dimensional representation of C_3. Then S^{λ_3} is a 2-dimensional sphere, where C_3 acts as a rotation by an angle of $\frac{2\pi}{3}$ around the z-axis. Here only 0 and ∞ are fixed.

Beware that one-point compactification does not define a functor $\mathbf{Mod}_{\mathbb{R}[G]} \longrightarrow G\mathbf{Top}$. The point is that a map $V \longrightarrow W$ of representations is proper if and only if it is injective, since a nontrivial kernel will not be compact. The conclusion is that we get a functor

$$S^{(-)}\colon \mathbf{Mod}^{\mathrm{inj}}_{\mathbb{R}[G]} \longrightarrow G\mathbf{Top}$$

[2]When I last taught MA551 in Fall 2017, this statement showed up as Problem 5 on Homework 9.

if we restrict to the subcategory of *injective* maps of representations.

We also have a notion of based G-space.

Definition 2.1.8. A **based G-space** is a G-space X equipped with an equivariant map $* \longrightarrow X$. This amounts to picking out a G-fixed point in X. A map of based G-spaces is just a basepoint-preserving map that is equivariant.

We denote by $G\mathbf{Top}_*$ the category of based G-spaces. Note that the functor $S^{(-)}$ in fact lands in $G\mathbf{Top}_*$, where we may choose either 0 or ∞ as the basepoint (it is more common to choose ∞).

2.1.1. *Restriction and induction.* Just like for representations, there are a number of change-of-groups constructions at our disposal.

The simplest is restriction. Definition 1.1.27 and Definition 1.1.32 work just as well here to define the restriction to a subgroup. We will again write $\downarrow_H^G(X)$ for the restriction. Note that this does not change the topology of the space at all, it merely restricts the action. Just as for representations, this construction has left and right adjoints.

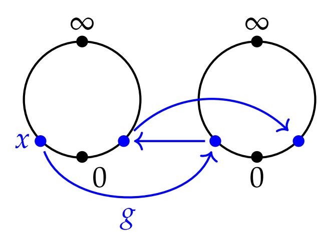
The analoge of Definition 1.1.35 is given here by the following construction.

Definition 2.1.9. Let $H \leq G$, and let X be an H-space. We define the **induced G-space** of X up to G, denoted either $\uparrow_H^G(X)$ or $G \times_H X$, as the quotient of $G \times X$ by the relation $(g \cdot h, x) \sim (g, h \cdot x)$ for all h, g in G and $x \in X$.

In particular, in the case of $H = e$, the induced G-space $\uparrow_e^G(X)$ is just the free G-space $G \times X$.

Example 2.1.10. For a subgroup $H \leq G$, the induced G-space $\uparrow_H^G(*)$ is $G \times_H * \cong G/H$. This will play an important role. More generally, if X is any space with a *trivial* H-action, then $\uparrow_H^G(X)$ will be $G/H \times X$, where G is acting on G/H (and trivially on X).

Example 2.1.11. Let $C_2 \leq C_4$ be the order two subgroup. Then the induced space $\uparrow_{C_2}^{C_4}(S^\sigma)$ is displayed to the right. Note, in particular, that it is not a sphere, but rather a disjoint union of spheres. I have displayed the action of the powers of the generator g of C_4 on a generic point x.



This example shows that the square of functors

$$\begin{tikzcd} \mathbf{Mod}_{\mathbb{R}[H]}^{\mathrm{inj}} \arrow[r, "\uparrow_H^G"] \arrow[d, "S^{(-)}"] & \mathbf{Mod}_{\mathbb{R}[G]}^{\mathrm{inj}} \arrow[d, "S^{(-)}"] \\ H\mathbf{Top} \arrow[r, "\uparrow_H^G"] & G\mathbf{Top} \end{tikzcd}$$

does **not** commute.

On the other hand, we have a direct analogue of Proposition 1.1.40, given as

Proposition 2.1.12. *Let $H \leq G$. Then induction of spaces is left adjoint to restriction:*

$$\uparrow_H^G \colon H\mathbf{Top} \rightleftarrows G\mathbf{Top} \colon \downarrow_H^G.$$

For example, in the case that $H = e$ is the trivial subgroup, this says that for a space X and G-space Y, a G-equivariant map $G \times X \longrightarrow Y$ is the same as a (nonequivariant) map $X \longrightarrow Y$.

$\boxed{\textbf{Mon, Oct. 19}}$

When we go to spectra later, we will be working in a based context. There is a based version of induction.

Definition 2.1.13. Let $H \le G$, and let X be a based H-space. We define the **induced based G-space** of X up to G, denoted also $\uparrow_H^G(X)$, as $G_+ \wedge_H X$, where this denotes the quotient of $G_+ \wedge X$ by the relation $(g \cdot h, x) \sim (g, h \cdot x)$ for all h, g in G and $x \in X$.

Note that $G_+ \wedge X$, which is sometimes called the "half smash product", can be identified with the quotient $(G \times X)/(G \times x_0)$, where $x_0 \in X$ is the basepoint. The analogue of Proposition 2.1.12 here is

Proposition 2.1.14. *Let $H \le G$. Then induction of spaces is left adjoint to restriction:*

$$\uparrow_H^G : H\mathbf{Top}_* \rightleftarrows G\mathbf{Top}_* : \downarrow_H^G.$$

Thus when we talk about a "free G-space" in the based context, we will always mean one of the form $G_+ \wedge X$, on which G acts freely except for the basepoint, which is G-fixed. Here the shearing isomorphism (see Worksheet 9) reads

$$G_+ \wedge_H X \cong (G/H)_+ \wedge X.$$

2.1.2. *Fixed points and orbit spaces.* Back in Section 1.1.9, we introduced the idea of the fixed points of a representation. The same definition works to define the fixed points of a G-space.

Definition 2.1.15. For $X \in G\mathbf{Top}$, we define the **fixed points** of X to be

$$X^G = \{x \in X \mid g \cdot x = x \text{ for all } g \in G\},$$

equipped with the subspace topology as a subspace of X.

Example 2.1.16. Recall the sign representation C_2-sphere S^σ from Example 2.1.5. As we indicated there, only the points 0 and ∞ are fixed. In other words, we have that $(S^\sigma)^{C_2} \cong S^0$.

Example 2.1.17. If we consider the regular representation C_2-sphere $S^{\rho_{C_2}}$, we find $(S^{\rho_{C_2}})^{C_2} \cong S^1$, because the regular representation has a one-dimensional trivial subrepresentation.

In complete analogy to Proposition 1.1.54, we have

Proposition 2.1.18. *The fixed points functor is right adjoint to the trivial G-action functor*

$$\mathbf{triv} = q^* : \mathbf{Top} \rightleftarrows G\mathbf{Top} : (-)^G.$$

Just as we did in the algebraic setting in Definition 1.1.55, we can define H-fixed points by first restricting down to an H-action and then passing to fixed points.

Definition 2.1.19. Given a G-space, we define the **H-fixed points** functor as the composite

$$G\mathbf{Top} \xrightarrow{\downarrow_H^G} H\mathbf{Top} \xrightarrow{(-)^H} \mathbf{Top}.$$

Example 2.1.20. Consider $G = K_4$. Recall the sign representation $p_1^*(\sigma)$. If we restrict this to either L or D, it becomes the sign representation σ, whereas if we restrict it to R, it is a trivial representation. This means that

$$(S^{p_1^*\sigma})^L \cong (S^{p_1^*\sigma})^D \cong S^0, \qquad (S^{p_1^*\sigma})^R \cong S^1.$$

By combining Proposition 2.1.12 and Proposition 2.1.18, we find that the H-fixed points functor is right adjoint to crossing with G/H:

(2.1.21)
$$\begin{tikzcd} \mathbf{Top} \arrow[r, shift left, "\mathbf{triv}"] \arrow[rr, bend left=40, "G/H\times"] & H\mathbf{Top} \arrow[l, shift left, "(-)^H"] \arrow[r, shift left, "\uparrow_H^G"] & G\mathbf{Top} \arrow[l, shift left, "\downarrow_H^G"] \arrow[ll, bend left=40, "(-)^H"] \end{tikzcd}$$

This even extends to a "topologically enriched" adjunction. What this means is that in each category we have not just a *set* of maps between any two objects but rather a *space* of maps. And a useful statement, which can be interpreted as part of the above adjunction, is that for $X \in G\mathbf{Top}$ we have a homeomorphism

(2.1.22)
$$\mathrm{Map}_G(G/H, X) \cong X^H,$$

where Map_G denotes the space of G-equivariant maps (this is a subspace of the space of all maps).

2.1.3. *Equivariant homotopy theory.* We have talked about G-spaces and equivariant maps, but as algebraic topologists we want to know the appropriate version of homotopy here.

Definition 2.1.23. Let $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ be equivariant maps between G-spaces. Letting G act on the interval I trivially, a **homotopy** between f and g is an equivariant map $h\colon I \times X \longrightarrow Y$ that restricts to f and g at times 0 and 1.

This looks identical to the usual notion. But note that since h is equivariant, this forces each map $h_t\colon X \longrightarrow Y$ to be equivariant. Thus h will be a homotopy *through equivariant maps*.

Example 2.1.24. Consider again the C_2-equivariant sign representation S^σ. Let $* \underset{0}{\overset{\infty}{\rightrightarrows}} S^\sigma$ be the inclusions of the two fixed points. Of course these two maps are homotopic if we forget about group actions, since S^σ is path-connected. But the two maps are not *equivariantly* homotopic, since the space of fixed points is not path-connected.

Now that we have a notion of homotopy, we also get the notion of homotopy equivalence, and also of homotopy groups. But if we consider (based) homotopy classes of maps out of S^n, considered as a trivial G-space, by Proposition 2.1.18 this will just compute $\pi_n(X^G)$. It turns out to be a good idea to also pay attention to the fixed points for subgroups as well. Since we have an (reverse) inclusion of fixed points $X^K \hookrightarrow X^H$ whenever $H \le K$, this yields a diagram of homotopy groups as displayed to the right. This looks like a Mackey functor, except that we don't have transfer maps. The data to the right is referred to as a **coefficient system** for G.

$$\begin{tikzcd} & \pi_n(X^G) \arrow[dl] \arrow[dr] & \\ \pi_n(X^H) \arrow[loop left] \arrow[dr] & & \pi_n(X^K) \arrow[loop right] \arrow[dl] \\ & \pi_n(X^e) \arrow[loop below] & \end{tikzcd}$$

Watch the video (in canvas): Coefficient systems for K_4.

Wed, Oct. 21

Last time, we introduced the homotopy G-coefficient systems $\underline{\pi}_n(X)$ associated to a (based) G-space X.

Definition 2.1.25. We say a G-equivariant map $f\colon X \longrightarrow Y$ is a **weak G-homotopy equivalence** if it induces an isomorphism on $\underline{\pi}_n$ for all $n \geq 0$. Strictly speaking, this means that it induces a bijection on $\underline{\pi}_0$ and that for each choice of basepoint $x_0 \in X^G$, the induced map $\underline{\pi}_n(X, x_0) \longrightarrow \underline{\pi}_n(Y, f(x_0))$ is an isomorphism.

Since the value of $\underline{\pi}_n$ at a subgroup H is computed by maps out of $(G/H)_+ \wedge S^n$, it follows that any G-homotopy equivalence is a weak G-homotopy equivalence. These two notions are equivalent on the class of equivariant CW complexes, which we now introduce.

Definition 2.1.26. A G-CW complex is a space X built as a colimit of G-spaces X_n, where X_0 is a discrete G-set and X_n is obtained from X_{n-1} by attaching cells of type $(G/H) \times D^n$ (H is not fixed).

Example 2.1.27. If X is a nonequivariant CW-complex, we may consider it as a G-CW complex with trivial G-action. Here all cells would be of type $(G/G) \times D^n$.

Example 2.1.28. Take $G = C_2$ and consider the sign representation sphere S^σ (see again the figure in Example 2.1.5). Here we start with the G-fixed 0-skeleton S^0. To this 0-skeleton, we attach a *single* free 1-cell, of type $(C_2/e) \times D^1$, where one (free) endpoint is attached to 0 and the other to the point ∞.

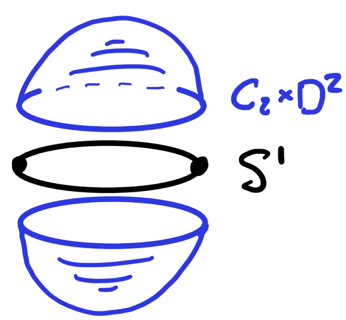
Example 2.1.29. Again take $G = C_2$ and consider the regular representation sphere. Here we have a one-dimensional fixed subsphere. We thus start with $X_1 = S^1_{\mathbf{triv}}$, with any ($G$-fixed) CW structure. We then attach a single free 2-cell. Here the equivariant attaching map

$$C_2 \times S^1 \longrightarrow S^1$$

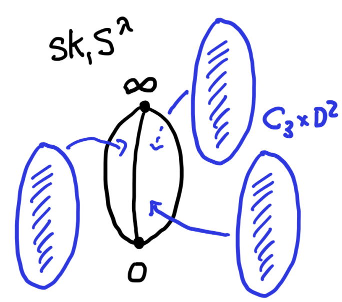
for the 2-cell corresponds to a nonequivariant map

$$S^1 \longrightarrow S^1,$$

which we take to be the identity.



Example 2.1.30. Take $G = C_3$ and consider the representation sphere S^λ_3. As mentioned in Example 2.1.7, only zero and ∞ are fixed, so we start with the G-fixed 0-skeleton S^0. Since there are no other fixed points, the higher-dimensional cells cannot be fixed (of type $C_3/C_3 \times D^n$) and so must be free (of type $C_3/e \times D^n$), since there are only two orbit types. If we attach a single free 1-cell, with endpoints attached to 0 and ∞, we get an "egg-beater", with three spokes. The last step is to fill in between the spokes. We do this by attaching a single free 2-cell.



Watch the video (in canvas): Cell structure on representation spheres for C_4

Like in non-equivariant topology, one of the main reasons to care about equivariant CW complexes is the following result.

Theorem 2.1.31 (Equivariant Whitehead Theorem)**.** *Let $f\colon X \longrightarrow Y$ be a weak G-homotopy equivalence between G-CW complexes. Then in fact f is a G-homotopy equivalence.*

Sketch. The idea is essentially the same as in the nonequivariant case (see, for example, Theorem 4.5 of Hatcher). We wish to produce an equivariant map $\varphi\colon Y \longrightarrow X$ that is an inverse-up-to-homotopy of f. Like most proofs involving CW complexes, the argument is by induction. We make the simplifying assumption that f is cellular (recall that the Cellular Approximation Theorem guarantees this is always possible up to homotopy, at least nonequivariantly).

The 0-skeleton of Y is a disjoint union of G-orbits, and we wish to define φ on each of these orbits. Thus let G/H be an orbit in the 0-skeleton of Y, and let use write y_0 for the point eH. Note that an equivariant map $G/H \longrightarrow X$ corresponds precisely to an H-fixed point of X (the image of y_0). By assumption, the map f is surjective on $\underline{\pi}_0$. In particular, there exists some x_0 in the 0-skeleton of X^H such that $f(x_0)$ lies in the same path-component of Y^H as y_0. We then set $\varphi(y_0) = x_0$ and let equivariance determine the values of φ on the rest of the orbit G/H.

By construction, the composition

$$\mathrm{sk}_0\, Y \xrightarrow{\varphi} \mathrm{sk}_0\, X \xrightarrow{f} \mathrm{sk}_0\, Y$$

is G-homotopic to the identity. Well, that is not literally true since the homotopy won't stay inside the 0-skeleton. What we really mean is that the above map, followed by the inclusion $\mathrm{sk}_0\, Y \hookrightarrow \mathrm{sk}_1\, Y$, is homotopic to the inclusion $\mathrm{sk}_0\, Y \hookrightarrow \mathrm{sk}_1\, Y$. On the other hand, injectivity of the induced map $f_*\colon \underline{\pi}_0(X) \longrightarrow \underline{\pi}_0(Y)$ shows that

$$\mathrm{sk}_0\, X \xrightarrow{f} \mathrm{sk}_0\, Y \xrightarrow{\varphi} \mathrm{sk}_0\, X$$

is also G-homotopic to the identity (after including into $\mathrm{sk}_1\, X$).

Next, let's work to define φ on the 1-skeleton of Y. We do this one 1-cell at a time. Thus suppose that $G/H \times S^0 \longrightarrow \mathrm{sk}_0\, Y$ is the attaching map for a 1-cell of Y. We wish to extend φ over this 1-cell. In other words, we wish to show that the composition

$$G/H \times S^0 \longrightarrow \mathrm{sk}_0\, Y \xrightarrow{\varphi} \mathrm{sk}_0\, X \hookrightarrow \mathrm{sk}_1\, X$$

can be extended to an equivariant map

$$G/H \times D^1 \longrightarrow \mathrm{sk}_1\, X.$$

By adjointness (2.1.21), this is equivalent to asking that the map

$$S^0 \longrightarrow (\mathrm{sk}_0\, Y)^H \xrightarrow{\varphi} (\mathrm{sk}_0\, X)^H \hookrightarrow (\mathrm{sk}_1\, X)^H$$

can be extended to an equivariant map

$$D^1 \longrightarrow (\mathrm{sk}_1\, X)^H.$$

The map $S^0 \longrightarrow (\mathrm{sk}_0\, Y)^H$ simply picks out two H-fixed points of Y, call them y_1 and y_2. The existence of the 1-cell in Y means that y_1 and y_2 lie in the same path-component of Y^H. Since $f_*\colon \pi_0(X^H) \longrightarrow \pi_0(Y^H)$ is injective, it follows that $f(y_1)$ and $f(y_2)$ lie in the same path-component of X^H, thereby allowing us to define the desired path $D^1 \longrightarrow (\mathrm{sk}_1\, X)^H$.

The fact that f induces an isomorphism on $\underline{\pi}_1$ (for each choice of basepoint) can then be used to show that $f \circ \varphi$ and $\varphi \circ f$ are homotopic to the identity (after including into the 2-skeleton to allow enough room for the homotopies). The same idea would be used to describe the induction step, in extending φ from the n-skeleton to the $(n+1)$-skeleton. ■

Mon, Oct. 26

2.1.4. *The Borel construction and Equivariant cohomology in the sense of Borel.* We are headed towards equivariant homology and cohomology, but I first want to take a detour to discuss another variant, which is often referred to as Borel equivariant cohomology. In fact, if you overhear someone talking about equivariant cohomology at a conference, most likely they are referring to this notion, but it is not as satisfactory as the notion of equivariant cohomology that we will discuss later.

Example 2.1.32. For any group G we will denote by EG a G-CW complex such that (1) the underlying space is contractible and (2) the action of G is free. We will denote the orbit space EG/G by BG. It is known as the **classifying space** for G.

Example 2.1.33. For $G = C_2$, we can take $EC_2 = S^\infty$, with the antipodal action of C_2. Just as the quotient of the antipodal action on S^n gives $\mathbb{RP}^n$, the quotient here gives $S^\infty/C_2 \cong \mathbb{RP}^\infty$. Thus $\boxed{BC_2 \simeq \mathbb{RP}^\infty.}$

Example 2.1.34. For $G = C_p$ with p an odd prime, we have a similar construction. Just as we can consider S^{2n-1} as the unit sphere in $\mathbb{C}^n$, we can similarly consider S^∞ as the unit sphere in $\mathbb{C}^\infty$. We can think of C_p as acting in each complex coordinate as multiplication by a pth root of unity. This gives a free action of C_p on S^∞, so we conclude that $EC_p \simeq S^\infty$ and $\boxed{BC_p \simeq S^\infty/C_p.}$ This odd-primary analogue of $\mathbb{RP}^\infty$ is known as an infinite-dimensional lens space.

Example 2.1.35. In fact, there was really no reason to restrict p to be prime in the example above. For any cyclic group C_n, the above construction works to give a model for $\boxed{BC_n \simeq S^\infty/C_n.}$

Example 2.1.36. Note that for any groups G and H, the product $EG \times EH$ satisfies the definition of $E(G \times H)$, and we find that $\boxed{B(G \times H) \simeq BG \times BH.}$ It follows that by taking products of the classifying space BC_n, we can get a classifying space for any finite abelian group.

In fact, there is a construction of EG, functorial in the group G, known as the bar complex. As we don't need it here, we will not go into the details.

Proposition 2.1.37. *The space BG is a $K(G,1)$ meaning that $\pi_1(BG) \cong G$ and all other homotopy groups vanish.*

Proof. The quotient map $EG \twoheadrightarrow BG$ is a universal cover with fibers given by G. This gives the fundamental group. The vanishing of the higher homotopy groups follows from the fact that covering maps induce isomorphisms on higher homotopy groups, together with the fact that EG is contractible. ■

The classifying space BG also relates group homology/cohomology by the next two results.

Proposition 2.1.38. *The cellular chains $C_*(EG)$ on EG form a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$.*

Proof. As the action of G on EG is free, it can only have G-free cells. Thus each such free cell contributes a copy of $\mathbb{Z}[G]$ if we consider $C_*(EG)$ as a chain complex of $\mathbb{Z}[G]$-modules. Since EG is contractible, it must have the same homology as a point, which is just $\mathbb{Z}$ in degree 0. ■

To see one construction of the G-space EG,

Watch the video (in canvas): The bar complex model for EG and BG and the bar resolution

Corollary 2.1.39. *For any abelian group A, thought of as a trivial G-module, we have isomorphisms*

$$\mathrm{H}_*(BG; A) \cong \mathrm{H}(G; A) \qquad \text{and} \qquad \mathrm{H}^*(BG; A) \cong \mathrm{H}^*(G).$$

So it turns out that the group homology and group cohomology we studied before were examples of homology and cohomology of spaces. In those previous contexts, we allowed nontrivial G-modules for coefficients, and in fact that is possible for the space BG as well, but this uses the idea of homology/cohomology with "local coefficients". But we will not discuss this here.

With Corollary 2.1.39 and Example 2.1.33, we can reinterpret Example 1.3.23 and Example 1.3.25 as giving

$$\mathrm{H}^*(\mathbb{RP}^\infty) \cong \mathbb{Z}[x_2]/(2x_2) \qquad \text{and} \qquad \mathrm{H}^*(\mathbb{RP}^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x_1].$$

Definition 2.1.40. Given a G-space X, we define the **Borel construction** on X to be the orbit space of the (diagonal) G-action on $EG \times X$. The Borel construction is often denoted $EG \times_G X$, and we define the Borel-equivariant homology and cohomology of X to be

$$\mathrm{H}_*^{\mathrm{Borel}}(X) = \mathrm{H}_*(EG \times_G X), \qquad \mathrm{H}^*_{\mathrm{Borel}}(X) = \mathrm{H}^*(EG \times_G X).$$

Borel cohomology is a kind of equivariant cohomology theory, but it is not the best one. We will soon discuss the more satisfactory Bredon cohomology theories. Nevertheless, as I mentioned at the start of the section, if you do a literature search for equivariant cohomology, most of your results are likely to concern Borel cohomology.

Example 2.1.41. There is a C_2-equivariant map $S^1_a \longrightarrow S^1$ that collapses the 0-skeleton onto the basepoint of S^1 and wraps around S^1 on each component of the free 1-cell $C_2 \times D_1$ of S^1_a. As a map of underlying spaces, this map $S^1_a \longrightarrow S^1$ is a degree two map. Thus if A is any abelian group on which multiplication by 2 is an isomorphism, it follows that the induced map

$$\mathrm{H}^*_{\mathrm{Borel}}(S^1; A) \longrightarrow \mathrm{H}^*_{\mathrm{Borel}}(S^1_a; A)$$

is an isomorphism. But these C_2-equivariant circles are certainly not equivariantly homotopy equivalent, as their fixed points spaces are not equivalent.

Remark 2.1.42. The Borel construction is also known as the **homotopy orbit space** and denoted X_{hG}. The reason is that we can think of $EG \longrightarrow *$ as a "free resolution". The ordinary orbit space X/G can be thought of as $* \times_G X$, and in the homotopy orbit space we have first "resolved" the point by EG before passing to orbits. We can do the same for fixed points. The fixed point space X^G can also be described as $\mathrm{Map}_G(*, X)$, and if we first replace the point by EG, we get $\mathrm{Map}_G(EG, X)$, which is known as the **homotopy fixed point space** and written X^{hG}.

Note that Borel cohomology can't tell the difference between $*$ and EG. More generally, we have

Definition 2.1.43. We say a G-equivariant map $f\colon X \longrightarrow Y$ is an **underlying equivalence** if it induces an equivalence on underlying spaces.

This is a rather course notion of equivalence, much weaker than G-homotopy equivalence. The primary example is $EG \longrightarrow *$, which is certainly not a G-homotopy equivalence, since the source has empty fixed points while the fixed points of the targer is a point, and in particular nonempty.

Proposition 2.1.44. *The functor $EG \times (-)\colon G\mathbf{Top} \longrightarrow G\mathbf{Top}$ takes underlying equivalences to G-equivalences. The homotopy orbit and homotopy fixed points constructions therefore take underlying equivalences to equivalences of spaces, and Borel cohomology takes underlying equivalences to isomorphisms.*

Wed, Oct. 28

2.1.5. *Borel cohomology as a Mackey functor.* We will now discuss how Borel cohomology determines a Mackey functor, and use this to prove Proposition 1.3.28.

First, note that if $H \leq G$ is a subgroup, then EG also serves as a model for EH. Then $EG/H \simeq BH$ and the quotient map

$$EG/H \twoheadrightarrow EG/G$$

provides a model for $BH \longrightarrow BG$ that is a covering map of degree $|G\colon H|$. It also follows that for any G-space X, the induced map on Borel constructions

$$p_H^G\colon EH \times_H X \longrightarrow EG \times_G X$$

is a covering map of degree $|G\colon H|$.

Now we recall that if $p\colon E \longrightarrow B$ is a covering map (of nonequivariant spaces), there is a "transfer map" in homology and cohomology

$$p^!\colon \mathrm{H}_*(B) \longrightarrow \mathrm{H}_*(E), \qquad p_!\colon \mathrm{H}^*(E) \longrightarrow \mathrm{H}^*(B).$$

One way to think of this map is that on the level of chains, it sends an n-chain on B to the sum of the lifts of B to E. Since the number of such lifts is the degree of the cover, we conclude that the composites

$$\mathrm{H}_*(B) \xrightarrow{p^!} \mathrm{H}_*(E) \xrightarrow{p_*} \mathrm{H}_*(B) \qquad \text{and} \qquad \mathrm{H}^*(B) \xrightarrow{p^*} \mathrm{H}^*(E) \xrightarrow{p_!} \mathrm{H}^*(B) \tag{2.1.45}$$

are multiplication by the degree of the cover.

Theorem 2.1.46. *For any G-space X and abelian group A of coefficients, the groups $\mathrm{H}^*_{\mathrm{Borel}}(EH \times_H X)$, as H ranges over the subgroups of G, form a G-Mackey functor.*

We can therefore write $\underline{\mathrm{H}}^*_{\mathrm{Borel}}(X)$ for this Mackey functor.

Sketch. We described above a (covering) map of spaces $p_H^G\colon EH \times_H X = EG \times_H X \twoheadrightarrow EG \times_G X$. We then define the restriction map as $(p_H^G)^*$. Similary, the transfer map is $(p_H^G)_!$. Next, we produce the conjugation homomorphism $\mathrm{H}^*(EH \times_H X) \xrightarrow{c_g} \mathrm{H}^*(EH^g \times_{H^g} X)$. This will be the map on cohomology induced by a map of spaces

$$EH^g \times_{H^g} X \longrightarrow EH \times_H X.$$

There are several ways to build this, but perhaps the simplest is to make use of the functorial construction (sketched in the video last time) of EH. Thus given a group homomorphism $K \xrightarrow{\varphi} G$, there is an induced map $EK \xrightarrow{E\varphi} EG$. Given a G-space X, we have an induced K-space $\varphi^*(X)$, where K acts via φ and the given G-action on X. All of this yields a map

$$EK \times_K \varphi^*(X) \xrightarrow{E\varphi \times \mathrm{id}} EG \times_G X.$$

Taking $K = H^g$ and $G = H$ and the homomorphism $\varphi\colon H^g \longrightarrow H$ of conjugation by g^{-1}, we get

$$EH^g \times_{H^g} \varphi^*(X) \longrightarrow EH \times_H X,$$

and it remains only to identity the H^g-spaces $\varphi^*(X)$ and X. The underlying space in both cases is X, and I leave you to verify that acting by g^{-1} provides an H^g-equivariant isomorphism $X \cong \varphi^*(X)$.

Most of the axioms from Definition 1.2.1 follow from analogous identities at the level of the maps of spaces. The axioms that take a little more work to verify are (1) the axiom that c_h is the identity on $\underline{\mathrm{H}}^*_{\mathrm{Borel}}(X)(H)$ and (2) the Double Coset Formula. The verification of (1) will be deferred to the worksheet, and we discuss (2) in a video.

Watch the video (in canvas): The Double Coset Formula for the Borel Mackey functor

■

Note that the property (2.1.45) implies that in the Borel Mackey functor, the composition

$$\underline{\mathrm{H}}^*_{\mathrm{Borel}}(X)(G) \xrightarrow{R^G_K} \underline{\mathrm{H}}^*_{\mathrm{Borel}}(X)(K) \xrightarrow{I^G_K} \underline{\mathrm{H}}^*_{\mathrm{Borel}}(X)(G)$$

is multiplication by the index $|G\colon K|$ (note that this is the *opposite* composition to what is considered for the Double Coset Formula). There is a name for Mackey functors with this property.

Definition 2.1.47. We say that $\underline{M} \in \mathbf{Mack}_G$ is a **cohomological Mackey functor** if, for each $H \leq K$, the composition

$$\underline{M}(K) \xrightarrow{R^K_H} \underline{M}(H) \xrightarrow{I^K_H} \underline{M}(K)$$

is multiplication by the index $|K\colon H|$.

The name comes from the fact that group cohomology satisfies this property. As we have discussed, the Borel Mackey functor $\underline{\mathrm{H}}^*_{\mathrm{Borel}}(X)$ is a cohomological Mackey functor.

With Theorem 2.1.46 now in hand, we return to prove Proposition 1.3.28.

Proof of Proposition 1.3.28. If p is prime and $S \leq G$ is a p-Sylow subgroup, then the index of S in G is prime to p. The restriction map listed in Proposition 1.3.28 can now be view as the map on cohomology induced by the covering map $q\colon BS \longrightarrow BG$. Moreover, by the above, the composition

$$\mathrm{H}^*(BG;\mathbb{F}_p) \xrightarrow{q^*} \mathrm{H}^*(BS;\mathbb{F}_p) \xrightarrow{q_!} \mathrm{H}^*(BG;\mathbb{F}_p)$$

is multiplication by $|G\colon S|$. This is an isomorphism, since we are using coefficients $\mathbb{F}_p$. It follows that the map q^* is injective.

Now, for *any* Mackey functor, the restriction $R^G_S\colon M(G) \longrightarrow M(S)$ always lands in the fixed points $M(S)^{W_G(S)}$ by the Weyl group action. It remains to show that the restriction *surjects* onto these fixed points if S is normal in G. Thus let $x \in \mathrm{H}^*(S;\mathbb{F}_p)^{W_G(S)}$ and let $y = I^G_S(x)$. Then the Double Coset formula tells us that

$$\begin{aligned} R^G_S(y) = R^G_S I^G_S(x) &= \sum_{SgS \in S\backslash G/S} I^S_{S^{g^{-1}}\cap S} c_{g^{-1}} R^S_{S\cap S^g}(x) \\ &= \sum_{gS\in G/S} c_{g^{-1}}(x) = |G\colon S| \cdot x. \end{aligned}$$

Again, since the index $|G\colon S|$ is invertible in $\mathbb{F}_p$, we are done. ■

Remark 2.1.48. It is also possible to describe the image of the restriction $\mathrm{H}^*(G) \longrightarrow \mathrm{H}^*(S)$ even if S is not normal. The answer is that it consists of classes $x \in \mathrm{H}^*(S)$ for which the diagram

$$\begin{tikzcd} \mathrm{H}^*(S) \arrow[rr, "c_g"] \arrow[dr, "R^S_{S\cap S^g}"'] & & \mathrm{H}^*(S^g) \arrow[dl, "R^{S^g}_{S\cap S^g}"] \\ & \mathrm{H}^*(S\cap S^g) & \end{tikzcd}$$

commutes for every $g \in G$. In the case that S is normal, this amounts just to the condition that x is in the Weyl-invariants.

$\boxed{\textbf{Mon, Nov. 2}}$

2.1.6. *Bredon homology and cohomology.* We have recently discussed Borel cohomology, defined as the ordinary cohomology of the Borel construction (also known as the homotopy orbit space). We now turn to our preferred version of equivariant homology and cohomology, due to Bredon.

We will start by defining the analogue of cellular chains in this context. First, we recall the notion of *coefficient system* from Section 2.1.3, now in a more formal way.

Definition 2.1.49. Let $\mathcal{O}_G$ denote the **orbit category** of G, whose objects are the G-orbits G/H and whose morphisms are just the G-equivariant maps.

Thus $\mathcal{O}_G$ is a full subcategory of the category $\mathbf{FinSet}_G$ of finite G-sets. Here, the set $\mathrm{Hom}_{\mathcal{O}_G}(G/H, G/K)$ is nonempty if and only if H is subconjugate to K (meaning that H is contained in some conjugate of K). Another relevant point is that the automorphism group of G/H in $\mathcal{O}_G$ is precisely the Weyl group $W_G(H)$.

Definition 2.1.50. A **coefficient system** for G is a functor $M\colon \mathcal{O}_G^{op} \longrightarrow \mathbf{AbGp}$.

We will use the same language in discussing coefficient systems that we used for Mackey functors. Thus if H is contained in K, we have a restriction map $R^K_H\colon M(G/K) \longrightarrow M(G/H)$. And we may abbreviate $M(G/H)$ to $M(H)$.

Example 2.1.51. Let $\underline{M}$ be a G-Mackey functor. Then by forgetting the data of transfer maps, we obtain a coefficient system. Strictly speaking, there is a little work to make the translation here. The first step is to note that a G-equivariant map $\varphi\colon G/H \twoheadrightarrow G/K$ can be factored (though not uniquely) as a composition $G/H \twoheadrightarrow G/K^\gamma \cong G/K$, where H is contained in the conjugate K^γ. Then the map

$$M(\varphi)\colon M(G/K) \longrightarrow M(G/H)$$

is defined to be

$$\underline{M}(K) \xrightarrow{c_\gamma} \underline{M}(K^\gamma) \xrightarrow{R^{K^\gamma}_H} \underline{M}(H).$$

You then have to show that this composition does not depend on the choice of factorization.

In particular, for any abelian group A, there is a constant coefficient system $\underline{A}$ in which all restriction and conjugation maps are the identity.

Example 2.1.52. We saw already that if $n \geq 2$ and X is a G-space, then $\underline{\pi}_n(X)(G/H) := \pi_n(X^H)$ defines a coefficient system.

What is really happening in this example is that the assignment $G/H \mapsto X^H$ defines a coefficient system of spaces, by (2.1.22), and we are then applying the functor π_n to get a coefficient system of abelian groups. We can just as well apply other functors to our coefficient system of spaces.

Example 2.1.53. Let X be a G-space, A an abeilan group, and $n \geq 0$. Then the assignment $\underline{\mathrm{H}}_n(X;A)(G/K) = \mathrm{H}_n(X^K;A)$ defines a coefficient system.

Example 2.1.54. Take $G = C_2$ and $X = S^\sigma$. Then $X^{C_2} \cong S^0$ and $X^e \cong S^1$. It follows that

$$\underline{\mathrm{H}}_0(S^\sigma;\mathbb{Z}) \cong \begin{array}{c} \mathbb{Z}^2 \\ \downarrow\nabla \\ \mathbb{Z}, \end{array} \qquad \underline{\mathrm{H}}_1(S^\sigma;\mathbb{Z}) \cong \begin{array}{c} 0 \\ \\ \mathbb{Z}_{\mathrm{sgn}}. \end{array}$$

Here we have written $\mathbb{Z}_{\mathrm{sgn}}$ for $\underline{\mathrm{H}}_1(S^\sigma)(C_2/e)$ since the nontrivial element of C_2 acts as an orientation-reversing map on S^σ.

Example 2.1.55. Let X be a G-CW complex and $n \geq 0$. Then the assignment $\underline{C}_n(X)(G/K) = C_n^{\text{cell}}(X^K)$ defines a coefficient system. This depends on the observation that the K-fixed points of a G-CW complex inherits a CW structure.

Example 2.1.56. Take $G = C_2$ and $X = S^\sigma$. Recall from Example 2.1.28 that this has a C_2-CW structure with two fixed 0-cells and a single free 1-cell. On the fixed points, this gives S^0, with two 0-cells and no higher cells, whereas on the underlying space it gives the cell structure for S^1 having two 0-cells and two 1-cells. Then

$$\underline{C}_0(S^\sigma;\mathbb{Z}) \cong \begin{array}{c}\mathbb{Z}^2 \\ \downarrow \mathrm{id} \\ \mathbb{Z}^2,\end{array} \qquad \underline{C}_1(S^\sigma;\mathbb{Z}) \cong \begin{array}{c} 0 \\ \\ \mathbb{Z}[C_2].\end{array}$$

Here we have written $\mathbb{Z}[C_2]$ for $\underline{C}_1(S^\sigma)(C_2/e)$ since the C_2-action exchanges the two 1-cells in $(S^\sigma)^e = S^1$.

Since coefficient systems are defined to be functors, there is an obvious notion of map between them.

Definition 2.1.57. A map of G-coefficient systems $C \longrightarrow D$ is a natural transformation of functors. We will write $\mathrm{Hom}_{\mathrm{Coeff}}(C, D)$ for the set of maps of coefficient systems (it has a natural structure of abelian group).

Thus this comprises maps $C(K) \longrightarrow D(K)$ for each $K \leq G$ that commute with restriction and conjugation homomorphisms.

Example 2.1.58. If we fix the subgroup K and allow n to vary in Example 2.1.55, then $\underline{C}_*(X)(G/K) = C_*^{\text{cell}}(X^K)$ is a chain complex. The differentials in these complexes commute with the restriction and conjugation homomorphisms, in the sense that we have commuting diagrams

$$\begin{array}{ccc} C_n^{\text{cell}}(X^K) & \xrightarrow{d_n} & C_{n-1}^{\text{cell}}(X^K) \\ \downarrow R_H^K & & \downarrow R_H^K \\ C_n^{\text{cell}}(X^H) & \xrightarrow{d_n} & C_{n-1}^{\text{cell}}(X^H). \end{array}$$

Thus we have maps $d_n \colon \underline{C}_n(X) \longrightarrow \underline{C}_{n-1}(X)$ of coefficient systems, which make $\underline{C}_*(X)$ into a *chain complex of coefficient systems*.

Watch the video (in canvas): Some examples of $\underline{C}_*(X)$.

Given the definitions we have thus far, it is in fact simpler to define Bredon cohomology before Bredon homology.

Definition 2.1.59. Let X be a G-space and M be a G-coefficient system. Then $\mathrm{Hom}_{\mathrm{Coeff}}(C_*(X), M)$ is a cochain complex of abelian groups, which we write $C^*_{\mathrm{Coeff}}(X; M)$, and we define the **Bredon cohomology of X with coefficients M** to be

$$\mathrm{H}^n_{\mathrm{Bredon}}(X; M) = \mathrm{H}^n\big(C^*_{\mathrm{Coeff}}(X; M)\big).$$

We will usually just write $\mathrm{H}^*_G(X; M)$ for Bredon cohomology.

We will start with examples next time.

Wed, Nov. 4

Last time, we introduced Bredon cohomology $H^*_G(X;M)$, where M is a G-coefficient system. We now look at some examples. As we will start with the choice of $M = \underline{\mathbb{Z}}$, it will be useful to have the following lemma.

Lemma 2.1.60. *Let C be a G-coefficient system. Then* $\mathrm{Hom}_{\mathrm{Coeff}}(C, \underline{\mathbb{Z}}) \cong \mathrm{Hom}_{\mathbf{AbGp}}(C(e)/G, \mathbb{Z})$.

Example 2.1.61. We start with $G = C_2$ and $X = S^\sigma$. In the video from last time, we found that $\underline{C}_*(S^\sigma)$ is

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{Z}^2 \arrow[d, "\mathrm{id}"] \\ \mathbb{Z}[C_2] \arrow[r, "\binom{1}{-1}"] & \mathbb{Z}^2 \end{tikzcd}$$

If we apply $\mathrm{Hom}_{\mathrm{Coeff}}(-, \underline{\mathbb{Z}})$, then Lemma 2.1.60 gives us

$$\mathbb{Z} \xleftarrow{\begin{pmatrix}1 & -1\end{pmatrix}} \mathbb{Z}^2.$$

We conclude that

$$H^k_{C_2}(S^\sigma; \underline{\mathbb{Z}}) \cong \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k > 0. \end{cases}.$$

Example 2.1.62. Next we consider $G = C_2$ and $X = S^{2\sigma}$. In the video from last time, we found that $\underline{C}_*(S^{2\sigma})$ is

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z}^2 \arrow[d, "\mathrm{id}"] \\ \mathbb{Z}[C_2] \arrow[r, "1-g"] & \mathbb{Z}[C_2] \arrow[r, "\binom{1}{-1}"] & \mathbb{Z}^2 \end{tikzcd}$$

If we apply $\mathrm{Hom}_{\mathrm{Coeff}}(-, \underline{\mathbb{Z}})$, then Lemma 2.1.60 gives us

$$\mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\begin{pmatrix}1 & -1\end{pmatrix}} \mathbb{Z}^2.$$

We conclude that

$$H^k_{C_2}(S^{2\sigma}; \underline{\mathbb{Z}}) \cong \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & k = 1 \text{ or } k > 2. \end{cases}.$$

Example 2.1.63. Now we consider $G = C_4$ and $X = S^\sigma$. In the video from last time, we found that $\underline{C}_*(S^\sigma)$ is given as in the diagram to the right. If we apply $\mathrm{Hom}_{\mathrm{Coeff}}(-, \underline{\mathbb{Z}})$, then Lemma 2.1.60 again gives us

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{Z}^2 \arrow[d, "\mathrm{id}"] \\ \mathbb{Z}[C_4/C_2] \arrow[r, "\binom{1}{-1}"] \arrow[d, "\mathrm{id}"] & \mathbb{Z}^2 \arrow[d, "\mathrm{id}"] \\ \mathbb{Z}[C_4/C_2] \arrow[r, "\binom{1}{-1}"] & \mathbb{Z}^2 \end{tikzcd}$$

$$\mathbb{Z} \xleftarrow{\begin{pmatrix}1 & -1\end{pmatrix}} \mathbb{Z}^2.$$

We conclude that

$$H^k_{C_4}(S^\sigma; \underline{\mathbb{Z}}) \cong \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k > 0. \end{cases}.$$

Example 2.1.64. Now we consider $G = C_4$ and $X = S^\lambda$. In the video from last time, we found that $\underline{C}_*(S^\lambda)$ is given as in the diagram to the right. If we apply $\mathrm{Hom}_{\mathrm{Coeff}}(-, \underline{\mathbb{Z}})$, then Lemma 2.1.60 again gives us

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z}^2 \arrow[d, "\mathrm{id}"] \\ 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z}^2 \arrow[d, "\mathrm{id}"] \\ \mathbb{Z}[C_4] \arrow[r, "1-g"] & \mathbb{Z}[C_4] \arrow[r, "\begin{pmatrix} 1 \\ -1 \end{pmatrix}"] & \mathbb{Z}^2 \end{tikzcd}$$

$$\mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\begin{pmatrix} 1 & -1 \end{pmatrix}} \mathbb{Z}^2.$$

We conclude that

$$\mathrm{H}^k_{C_4}(S^\lambda; \underline{\mathbb{Z}}) \cong \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & k = 1 \text{ or } k > 2. \end{cases}.$$

In fact, we can deduce from Lemma 2.1.60 that cohomology with coefficients in $\underline{\mathbb{Z}}$ is just the cohomology of the orbit spce.

Proposition 2.1.65. *For any G-space X, we have* $\boxed{\mathrm{H}^n_G(X; \underline{\mathbb{Z}}) \cong \mathrm{H}^n(X/G; \mathbb{Z}).}$

The orbit space in Example 2.1.61 and Example 2.1.63 is $I \simeq *$. And the orbit space in Example 2.1.62 and Example 2.1.64 is equivalent to S^2 (we have attached a two-cell to a contractible 1-skeleton).

All of the above have used $\underline{\mathbb{Z}}$ for coefficients. Another choice is to use the inflated Mackey functor $\mathrm{Inf}^G_{G/G}(\mathbb{Z})$ (see Definition 1.2.30), which is nonzero only at the fixed point level.

Lemma 2.1.66. *Let C be a G-coefficient system. Then* $\mathrm{Hom}_{\mathrm{Coeff}}(C, \mathrm{Inf}^G_{G/G}(\mathbb{Z})) \cong \mathrm{Hom}_{\mathbf{AbGp}}(C(G), \mathbb{Z})$.

It follows in all of the above examples, the dual of the top row in $\underline{C}_*(X)$ will give $\underline{C}^*(X; \mathrm{Inf}^G_{G/G}(\mathbb{Z}))$. In each of these examples, the only nontrivial cohomology group is just $\mathbb{Z}^2$ in degree zero. In fact, we see that in general the following is true.

Proposition 2.1.67. *For any G-space X, we have* $\boxed{\mathrm{H}^n_G(X; \mathrm{Inf}^G_{G/G}(\mathbb{Z})) \cong \mathrm{H}^n(X^G; \mathbb{Z}).}$

We are getting the cohomology of the fixed points as one example of Bredon cohomology. Another choice of coefficients will recover the cohomology of the underlying space.

Lemma 2.1.68. *Let C be a G-coefficient system. Then* $\mathrm{Hom}_{\mathrm{Coeff}}(C, \uparrow^G_e(\mathbb{Z})) \cong \mathrm{Hom}_{\mathbf{AbGp}}(C(e), \mathbb{Z})$.

This lemma is really a manifestation of Proposition 1.1.46. The lemma then implies the following result.

Proposition 2.1.69. *For any G-space X, we have* $\boxed{\mathrm{H}^n_G(X; \uparrow^G_e(\mathbb{Z})) \cong \mathrm{H}^n(X; \mathbb{Z}).}$

Let's look at one more choice of coefficients. For $G = C_p$, let $\underline{F} = \ker(\underline{\mathbb{Z}} \longrightarrow \mathrm{Inf}(\mathbb{Z}))$. Thus $\underline{F} = \begin{matrix} 0 \\ \downarrow \\ \mathbb{Z} \end{matrix}$. The short exact sequence $\underline{F} \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \mathrm{Inf}(\mathbb{Z})$ gives rise to a long exact sequence in cohomology

$$\cdots \longrightarrow \mathrm{H}^n_{C_p}(X; \underline{F}) \longrightarrow \mathrm{H}^n_{C_p}(X; \underline{\mathbb{Z}}) \longrightarrow \mathrm{H}^n_{C_p}(X; \mathrm{Inf}(\mathbb{Z})) \longrightarrow \cdots.$$

By Proposition 2.1.65 and Proposition 2.1.67, this can be rewritten as

$$\cdots \longrightarrow \mathrm{H}^n_{C_p}(X; \underline{F}) \longrightarrow \mathrm{H}^n(X/C_p; \mathbb{Z}) \longrightarrow \mathrm{H}^n(X^{C_p}; \mathbb{Z}) \longrightarrow \cdots.$$

As X^{C_p} is a subspace on which C_p acts trivially, we can identify X^{C_p} with a subspace of X/C_p. From the above long exact sequence, we deduce the following result.

Proposition 2.1.70. *For any C_p-space X, we have* $\boxed{\mathrm{H}^n_{C_p}(X; \underline{F}) \cong \mathrm{H}^n(X/C_p, X^{C_p}; \mathbb{Z}) \cong \widetilde{\mathrm{H}}^n((X/X^{C_p})/C_p; \mathbb{Z}).}$

Mon, Nov. 9

Having discussed Bredon cohomology last week, we now turn to Bredon homology. Recall from Example 2.1.58 that we have a chain complex $\underline{C}_*(X)$ of G-coefficient systems for any G-space X. To define cohomology with coefficients in a coefficient system M, we applied $\mathrm{Hom}_{\mathrm{Coeff}}(-, M)$ to get a cochain complex. This makes sense since $\underline{C}_n(X)$ and M are the same type of object (namely, a G-coefficient system). For homology, we would expect to instead consider something like $\underline{C}_*(X) \otimes_{\mathrm{Coeff}} M$. The problem is that this does not make sense. Remember that for modules over a non-commutative ring, we would take a tensor product of a right module with a left module. Here, since $\underline{C}_n(X)$ is a contravariant functor on the orbit category $\mathcal{O}_G$, we would want to "tensor" $\underline{C}_n(X)$ with a *covariant* functor.

Definition 2.1.71. Let C be a G-coefficient system and N be a covariant functor $\mathcal{O}_G \xrightarrow{N} \mathbf{AbGp}$. We then define $C \otimes_{\mathcal{O}_G} N \in \mathbf{AbGp}$ as the quotient

$$C \otimes_{\mathcal{O}_G} N = \Big(\bigoplus_{H \leq G} C(H) \otimes N(H) \Big) / \sim,$$

where

(1) $r(x) \otimes y \sim x \otimes t(y)$ for $x \in C(K)$, $y \in N(H)$, and $H \leq K$ and
(2) $c_g(x) \otimes y \sim x \otimes c_g(y)$ for $x \in C(K^g)$ and $y \in N(K)$.

Remark 2.1.72. I prefer to think of this as the coequalizer of

$$\bigoplus_{H,K \leq G} \; \bigoplus_{\mathcal{O}(G/H, G/K)} C(K) \otimes N(H) \rightrightarrows \bigoplus_{H \leq G} C(H) \otimes N(H),$$

where one arrow uses the contravariant functoriality of C and the other uses the covariant functoriality of N.

Because of the above point of view, and in order to distinguish covariant functors from contravariant functors, I will refer to covariant functors $\mathcal{O}_G \longrightarrow \mathbf{AbGp}$ as $\mathcal{O}_G$**-modules**.

Definition 2.1.73. Let X be a G-space and N be an $\mathcal{O}_G$-module. We then define the **Bredon homology** of X with coefficients in N to be

$$\mathrm{H}_n^{\mathrm{Bredon}}(X; N) = \mathrm{H}_n(\underline{C}_*(X) \otimes_{\mathcal{O}_G} N).$$

We will usually just write $\mathrm{H}_n^G(X; N)$ for Bredon homology.

Like the case for coefficient systems, a good source of $\mathcal{O}_G$-modules is Mackey functors. If $\underline{M}$ is a Mackey functor, we obtain an $\mathcal{O}_G$-module by forgetting the information of restriction maps.

Remark 2.1.74. There is one additional subtlety when it comes to the conjugation morphisms in a Mackey functor $\underline{M}$. Consider an isomorphism $G/H \cong G/H^\gamma$ in $\mathcal{O}_G$. According to Worksheet 12, such an isomorphism is given by the formula $gH \mapsto g \cdot \gamma^{-1} H^\gamma$. This should produce maps of opposite variance in a coefficient system and an $\mathcal{O}_G$-module. For the coefficient system, we use the map $c_{\gamma^{-1}} \colon \underline{M}(H^\gamma) \longrightarrow \underline{M}(H)$. In contrast, for the $\mathcal{O}_G$-module, we use $c_g \colon \underline{M}(H) \longrightarrow \underline{M}(H^\gamma)$.

Similarly to Lemma 2.1.60, we have

Lemma 2.1.75. *Let C be a G-coefficient system. Then*

(1) $C \otimes \underline{\mathbb{Z}} \cong C(e)/G$,
(2) $C \otimes \mathrm{Inf}_{G/G}^G(\mathbb{Z}) \cong C(G)$, *and*
(3) $C \otimes \uparrow_e^G (\mathbb{Z}) \cong C(e)$.

This leads to the following result.

Corollary 2.1.76. *Let X be a G-space. Then*

(1) $\mathrm{H}_n^G(X;\underline{\mathbb{Z}}) \cong \mathrm{H}_n(X/G;\mathbb{Z})$,
(2) $\mathrm{H}_n^G(X;\mathrm{Inf}_{G/G}^G(\mathbb{Z})) \cong \mathrm{H}_n(X^G;\mathbb{Z})$, *and*
(3) $\mathrm{H}_n^G(X;\uparrow_e^G(\mathbb{Z})) \cong \mathrm{H}_n(X;\mathbb{Z})$.

These results tell us that Bredon homology and cohomology are quite general and all-encompassing. For instance, we get the much more narrow Borel cohomology as a special case:

$$\mathrm{H}^*_{\mathrm{Borel}}(X;\mathbb{Z}) = \mathrm{H}^*(EG\times_G X;\mathbb{Z}) \cong \mathrm{H}^*_{\mathrm{Bredon}}(EG\times X;\underline{\mathbb{Z}}),$$
$$\mathrm{H}_*^{\mathrm{Borel}}(X;\mathbb{Z}) = \mathrm{H}_*(EG\times_G X;\mathbb{Z}) \cong \mathrm{H}_*^{\mathrm{Bredon}}(EG\times X;\underline{\mathbb{Z}})$$

The natural projection $EG\times X \longrightarrow X$ therefore induces a natural map of graded rings

$$\mathrm{H}^*_{\mathrm{Bredon}}(X;\underline{\mathbb{Z}}) \longrightarrow \mathrm{H}^*_{\mathrm{Borel}}(X;\mathbb{Z}). \tag{2.1.77}$$

In the case of $X=*$, this is just the unit map

$$\mathbb{Z}\cong \mathrm{H}^*_{\mathrm{Bredon}}(*;\underline{\mathbb{Z}}) \longrightarrow \mathrm{H}^*_{\mathrm{Borel}}(*;\mathbb{Z}) \cong \mathrm{H}^*(BG;\mathbb{Z}) \cong \mathrm{H}^*(G;\mathbb{Z})$$

for the cohomology ring of G. Another interesting case is when X is a free G-space.

Proposition 2.1.78. *Suppose that X is a G-CW complex with only free cells, then* (2.1.77) *is an isomorphism.*

Proof. The point is that if X is G-free, then the projection $EG\times X\longrightarrow X$ is a G-equivariant weak homotopy equivalence (Definition 2.1.25). Since X was assumed to be G-CW, the Whitehead Theorem (Theorem 2.1.31) implies that the projection is a G-equivariant homotopy equivalence and therefore induces an isomorphism in cohomology. ∎

Example 2.1.79. Consider $G=C_2$. As discussed in Example 2.1.41, we have an equivariant map $S^1_a \longrightarrow S^1$ from the antipodal circle to the G-fixed circle, which is a double cover on underlying spaces.

Watch the video (in canvas): From Bredon to Borel

Example 2.1.80. Consider $G=C_2$ and $X=S^\sigma$. Another way to describe the cell structure given in Example 2.1.28 is to say that we have an equivariant cofiber sequence

$$(C_2)_+ \longrightarrow S^0 \longrightarrow S^\sigma.$$

This cofiber sequence induces long exact sequences in Borel and Bredon cohomology.

$$\begin{tikzcd}
\widetilde{\mathrm{H}}^*_{\mathrm{Borel}}(S^\sigma;\mathbb{Z}) \arrow[r] \arrow[d] & \widetilde{\mathrm{H}}^*_{\mathrm{Borel}}(S^0;\mathbb{Z}) \arrow[r] \arrow[d] & \widetilde{\mathrm{H}}^*_{\mathrm{Borel}}((C_2)_+;\mathbb{Z}) \arrow[d, "\cong"] \\
\widetilde{\mathrm{H}}^*_{\mathrm{Bredon}}(S^\sigma;\underline{\mathbb{Z}}) \arrow[r] & \widetilde{\mathrm{H}}^*_{\mathrm{Bredon}}(S^0;\underline{\mathbb{Z}}) \arrow[r, "\cong"] & \widetilde{\mathrm{H}}^*_{\mathrm{Bredon}}((C_2)_+;\underline{\mathbb{Z}}).
\end{tikzcd}$$

This becomes

$$\begin{tikzcd}
\widetilde{\mathrm{H}}^*_{\mathrm{Borel}}(S^\sigma;\mathbb{Z}) \arrow[r] \arrow[d] & \mathbb{Z}[x_2]/2x_2 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d, "\cong"] \\
\widetilde{\mathrm{H}}^*_{\mathrm{Bredon}}(S^\sigma;\underline{\mathbb{Z}}) \arrow[r] & \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z}.
\end{tikzcd}$$

As in Example 2.1.61, we see that $\widetilde{\mathrm{H}}^*_{\mathrm{Bredon}}(S^\sigma;\underline{\mathbb{Z}})=0$. On the other hand, we conclude from the exact sequence that $\widetilde{\mathrm{H}}^*_{\mathrm{Borel}}(S^\sigma;\mathbb{Z})$ is the ideal $(x_2)\subset\mathbb{Z}[x_2]/2x_2$. In other words,

$$\mathrm{H}^*_{\mathrm{Borel}}(S^\sigma;\mathbb{Z})\cong\mathbb{Z}[x_2]/2x_2.$$

Wed, Nov. 11

2.1.7. *Application: Smith theory.* One application of Bredon cohomology is a simple proof of the following result.

Theorem 2.1.81 (Smith)**.** *Let G be a finite p-group and X be a finite G-CW complex. Suppose that the $\mathrm{H}^*(X;\mathbb{F}_p)$ looks like that of a sphere. Then either X^G is empty or $\mathrm{H}^*(X^G;\mathbb{F}_p)$ looks like that of a sphere.*

Note that the cohomology groups in the statement are all nonequivariant.

Sketch. The first step is to reduce to the case of $G = C_p$. This uses the fact that if G is a p-group of order larger than p, then G has a nontrivial proper normal subgroup $N \trianglelefteq G$. By induction, suppose that the result holds for groups of order less than that of G. But now if X is a G-CW complex, then $X^G \cong (X^N)^{G/N}$. The result now follows since both N and G/N have order small than G.

So now we consider the case of $G = C_p$. For simplicity, we suppose that $p = 2$. For the case of p odd, see [M, Section IV.1] or [B, Section 1.5]. We start by noting that the short exact sequence

$$\mathbb{F}_2 \xrightarrow{\Delta} \mathbb{F}_2[C_2] \xrightarrow{\nabla} \mathbb{F}_2$$

extends to a short exact sequence of coefficient systems

$$\underline{f} \hookrightarrow \uparrow_e^{C_2}(\mathbb{F}_2) \longrightarrow \underline{f} \oplus \mathrm{Inf}(\mathbb{F}_2),$$

where $\underline{f}$ is the $\mathbb{F}_2$-analogue of the $\underline{F}$ described below Proposition 2.1.69. There results a long exact sequence in Bredon cohomology

$$\mathrm{H}^n_{C_2}(X;\underline{f}) \longrightarrow \mathrm{H}^n_{C_2}(X;\uparrow_e^{C_2}(\mathbb{F}_2)) \longrightarrow \mathrm{H}^n_{C_2}(X;\underline{f}) \oplus \mathrm{H}^n_{C_2}(X;\mathrm{Inf}(\mathbb{F}_2)) \longrightarrow \mathrm{H}^{n+1}_{C_2}(X;\underline{f})$$

According to the $\mathbb{F}_2$-analogues of Proposition 2.1.70, Proposition 2.1.69, and Proposition 2.1.67, this long exact sequence can be rewritten as

$$\widetilde{\mathrm{H}}^n((X/X^{C_2})/C_2;\mathbb{F}_2) \longrightarrow \mathrm{H}^n(X;\mathbb{F}_2) \longrightarrow \widetilde{\mathrm{H}}^n((X/X^{C_2})/C_2;\mathbb{F}_2) \oplus \mathrm{H}^n(X^{C_2};\mathbb{F}_2) \longrightarrow \widetilde{\mathrm{H}}^{n+1}((X/X^{C_2})C_2;\mathbb{F}_2).$$

Letting

$$a_n = \dim \widetilde{\mathrm{H}}^n((X/X^{C_2})/C_2;\mathbb{F}_2), \qquad b_n = \dim \mathrm{H}^n(X;\mathbb{F}_2), \qquad \text{and} \quad c_n = \dim \mathrm{H}^n(X^{C_2};\mathbb{F}_2),$$

exactness (at the direct sum) in the long exact sequence implies that

$$a_n + c_n \le b_n + a_{n+1}.$$

This then gives that

$$a_0 + c_0 + c_1 \le b_0 + a_1 + c_1 \le b_0 + b_1 + a_2,$$

or more generally that

$$a_0 + c_0 + c_1 + \cdots + c_n \le b_0 + b_1 + \cdots + b_n + a_{n+1}$$

for any n. Since X was assumed to be a finite complex, we know that a_{n+1} vanishes for n large enough. We conclude that $\sum c_i \le \sum b_i$. But now by assumption, we know that $\sum b_i = 2$. It follows that $\sum c_i \le 2$. If we assume that X^{C_2} is nonempty, this tells us that $\sum c_i > 0$, so it remains to eliminate the possibility that $\sum c_i = 1$. But the long exact sequence also tells us that

$$\chi(X) = \chi(X^{C_2}) + 2\chi((X/X^{C_2})/C_2) \equiv \chi(X^{C_2}) \pmod 2.$$

This finishes the proof. ∎

2.1.8. *Axioms for equivariant homology and cohomology.* By the category of pairs of G-CW complexes, we mean the category in which the objects are a pair (X, A), where X is G-CW and A is a subcomplex, and a morphism $f : (X, A) \longrightarrow (Y, B)$ is a map $f : X \longrightarrow Y$ such that $f(A) \subseteq B$.

Definition 2.1.82. A **homology theory** on G-CW complexes is a sequence of functors $h_n(X, A)$ on pairs of G-CW complexes and natural transformations $\delta : h_n(X, A) \longrightarrow h_{n-1}(A, \emptyset)$ satisfying the following axioms:

(1) (Homotopy) If $f \simeq g$, then $f_* = g_*$
(2) (Long exact sequence) Writing $h_n(X) := h_n(X, \emptyset)$, the sequence

$$\dots h_n(A) \longrightarrow h_n(X) \longrightarrow h_n(X, A) \xrightarrow{\delta} h_{n-1}(A) \longrightarrow \dots$$

is exact
(3) (Excision) If X is the union of subcomplexes A and B, then the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism

$$h_n(A, A \cap B) \cong h_n(X, B)$$

(4) (Additivity) If (X, A) is the disjoint union of pairs (X_i, A_i), then the inclusions $(X_i, A_i) \longrightarrow (X, A)$ induce an isomorphism

$$\bigoplus_i h_n(X_i, A_i) \cong h_n(X, A).$$

An **ordinary homology theory** is one that also satisfies the additional axiom

(5) (Dimension) $h_n(G/H) = 0$ if $n \neq 0$.

Note that since we have an inclusion of $\mathcal{O}_G$ into the category of G-CW complexes, a homology theory h_* determines an $\mathcal{O}_G$-module by the formula $M(G/H) = h_0(G/H)$. It turns out that if h is an ordinary homology theory and M is the resulting $\mathcal{O}_G$-module, then we have an isomorphism $h_n(X, A) \cong \mathrm{H}_n^{\mathrm{Bredon}}(X, A; M)$. In other words, Bredon homology is essentially the only ordinary equivariant homology theory. Just like the nonequivariant situation, there are many "extraordinary" equivariant homologies theories as well.

The axioms for cohomology are the same, except that the variance is reversed (cohomology is contravariant).

2.2. **Group actions in stable homotopy theory: G-spectra.**

Another point of view on nonequivariant homology and cohomology theories is that of *stable homotopy theory*.

2.2.1. *The nonequivariant stable homotopy category.* One way to think of the category of spectra is that it is the result of starting with the category $\mathbf{Top}_*$ of based spaces and "inverting" the circle S^1. This is not meant to be a precise statement, but nevertheless it is helpful to think of this as an equivalence

$$\boxed{\mathbf{Ho}(\mathbf{Sp}) \simeq \mathbf{Ho}(\mathbf{Top}_*)[(S^1)^{-1}]}\,.$$

The homotopy category $\mathbf{Ho}(\mathbf{Sp})$ is known as the **stable homotopy category**. A motivation for this is the **Freudenthal Suspension Theorem**, which says that for any based space X and $n \geq 0$, the suspension maps

$$\pi_n(X) \longrightarrow \pi_{n+1}(\Sigma X) \longrightarrow \pi_{n+2}(\Sigma^2 X) \longrightarrow \cdots$$

eventually stabilize (meaning that after finitely many steps, the maps all become isomorphisms). These stable values are the so-called **stable homotopy groups** of X and tend to be easier to compute than the ordinary (unstable) homotopy groups.

We might ask for some object whose nth homotopy group is the nth *stable* homotopy group of X, and that is precisely what spectra give us.

Definition 2.2.1. A **spectrum** is a sequence $\{E_n\}$ of based spaces and maps $\sigma_n \colon \Sigma E_n \longrightarrow E_{n+1}$.

Example 2.2.2. For any based space X, the **suspension spectrum** of X, denoted $\Sigma^\infty X$, is the sequence $\{E_n = \Sigma^n X\}$, together with the identifcations $\Sigma E_n = \Sigma\Sigma^n X \cong \Sigma^{n+1} X = E_{n+1}$ for the maps σ_n.

The suspension spectrum construction defines a functor

$$\Sigma^\infty \colon \mathbf{Top}_* \longrightarrow \mathbf{Sp},$$

which descends to an induced functor

$$\Sigma^\infty \colon \mathbf{Ho}(\mathbf{Top}_*) \longrightarrow \mathbf{Ho}(\mathbf{Sp}).$$

We write $\mathbf{S}^0$ for $\Sigma^\infty S^0$. This is called the **sphere spectrum**. Similarly, we write $\mathbf{S}^n$ for $\Sigma^\infty S^n$. In fact, in the world of spectra, we can also make sense of $\mathbf{S}^n$ when n is negative. For example, the spectrum $\mathbf{S}^{-1}$ is the sequence

$$\{*, S^0, S^1, S^2, \dots\}, \qquad \Sigma * \xrightarrow{\sigma_0} S^0, \qquad \Sigma S^0 \xrightarrow{\sigma_1} S^1, \qquad \Sigma S^1 \xrightarrow{\sigma_2} S^2, \dots$$

where σ_0 is the constant map at the basepoint, but the higher σ_n's look like those for $\mathbf{S}^0$, but just shifted by one in the sequencing. Similarly, $\mathbf{S}^{-2}$ would be

$$\{*, *, S^0, S^1, \dots\}, \qquad \Sigma * \xrightarrow{\sigma_0} *, \qquad \Sigma * \xrightarrow{\sigma_1} S^0, \qquad \Sigma S^0 \xrightarrow{\sigma_2} S^1, \dots$$

Now for any spectrum E, we can define the homotopy groups by the formula

$$\pi_n(E) = [\mathbf{S}^n, E]_{\mathbf{Ho}(\mathbf{Sp})}.$$

Again, this is now defined even for n negative.

One of the most important facts about the stable category is that it is the appropriate home for cohomology theories.

Theorem 2.2.3 (Brown Representability)**.** *Let h^* be a cohomology theory on CW complexes. Then there exists a spectrum E and a natural isomorphism*

$$h^n(X) \cong [\mathbf{S}^{-n} \wedge \Sigma^\infty X, E]_{\mathbf{Ho}(\mathbf{Sp})}.$$

I have not described the smash product of spectra (that is a complicated story), but in the case of smashing with $\mathbf{S}^{-n}$, you can just think of it as shifting the sequence of spaces by n spots, just as $\mathbf{S}^{-n}$ is the n-fold shift of the sequence for $\mathbf{S}^0$.

For example, in the case of ordinary cohomology with coefficients in a group M, the representing spectrum is the **Eilenberg-Mac Lane spectrum** HM whose nth space is the Eilenberg-Mac Lane space $K(M, n)$.

$\boxed{\textbf{Mon, Nov. 16}}$

2.2.2. *The equivariant stable homotopy category, take 1.* Following the discussion in Section 2.2.1, one guess for the equivariant stable homotopy category is that we should start with the category of G-spaces and invert the circle S^1.

Definition 2.2.4. A G-equivariant S^1-spectrum will mean a sequence $\{E_n\}$ of based G-spaces and G-equivariant structure maps $\sigma_n \colon \Sigma E_n \longrightarrow E_{n+1}$ for all n.

We will write $\mathbf{Sp}^G_{S^1}$ for the category of G-equivariant S^1-spectra. We then have

$$\boxed{\mathbf{Ho}(\mathbf{Sp}^G_{S^1}) \simeq \mathbf{Ho}(G\mathbf{Top}_*)[(S^1)^{-1}]\,.}$$

In the literature, these G-spectra are sometimes referred to as **"naive" G-spectra**. We also have suspension spectra in this context:

Example 2.2.5. For any based G-space X, the **equivariant suspension spectrum** of X, denoted $\Sigma^\infty_G X$, is the sequence $\{E_n = \Sigma^n X\}$, together with the identifcations $\Sigma E_n = \Sigma\Sigma^n X \cong \Sigma^{n+1} X = E_{n+1}$ for the maps σ_n. Note that here G is acting trivially on the suspension coordinates, but (possibly) nontrivially on X.

This defines functors

$$\Sigma^\infty_G \colon G\mathbf{Top}_* \longrightarrow \mathbf{Sp}^G_{S^1} \qquad \text{and} \qquad \Sigma^\infty_G \colon \mathbf{Ho}(G\mathbf{Top}_*) \longrightarrow \mathbf{Ho}(\mathbf{Sp}^G_{S^1}).$$

Again, we write $\mathbf{S}^n_G$ for $\Sigma^\infty_G S^n$, and we can make sense of $\mathbf{S}^n_G$ for n negative by shifting our sequence to the right, just as we did in Section 2.2.1.

This category of G-spectra (we will soon discuss a different category of G-spectra) is good enough to represent cohomology theories:

Theorem 2.2.6 (Equivariant Brown Representability)**.** *Let h^* be an equivariant cohomology theory on G-CW complexes. Then there exists a G-equivariant S^1-spectrum E and a natural isomorphism*

$$h^n(X) \cong [\mathbf{S}^{-n} \wedge \Sigma^\infty_G X, E]_{\mathbf{Ho}(\mathbf{Sp}^G_{S^1})}.$$

Example 2.2.7. Let h^* be an *ordinary* cohomology theory, and let M be the coefficient system defined by $M(G/H) = h^0(G/H)$. Then this cohomology theory is represented by the **equivariant Eilenberg-Mac Lane spectrum** $H_G M$ whose nth space is an Eilenberg-Mac Lane space $K(M,n)$ satisfying

$$\underline{\pi}_j(K(M,n)) \cong \begin{cases} M & j = n \\ 0 & \text{else.} \end{cases}$$

Recall from Definition 2.1.25 that $\underline{\pi}_j(X)$ is the coefficient system defined by

$$\underline{\pi}_j(X)(G/H) = \pi_j(X^H).$$

Ok, so why did I say this is only "take 1" for the equivariant stable homotopy category? In other words, what is deficient about these equivariant cohomology theories? Well, one of the important theorems in nonequivariant cohomology is Poincaré duality. Recall that this says that if M is an orientable, compact n-manifold, then there is an isomorphism

$$\mathrm{H}_k(M;\mathbb{Z}) \cong \mathrm{H}^{n-k}(M;\mathbb{Z})$$

for every $k \geq 0$. This will not work equivariantly. Part of the reason is that in proofs of duality, one uses an embedding of the manifold M into some Euclidean space $\mathbb{R}^n$. Equivariantly, this will not be possible unless the Euclidean space is given a G-action. So representations enter into the theory on their own. The above version of G-spectra, and the corresponding cohomology theories, fail to account for nontrivial G-actions on Euclidean space, in other words, G-representations.

2.2.3. *The equivariant stable homotopy category, take 2.* Another guess for the equivariant stable homotopy category is that we should start with the category of G-spaces and invert the representation spheres S^V. Since any representation sits inside a direct sum of copies of the regular representation ρ_G, it is sufficient to invert ρ_G. Or we can invert all finite-dimensional representations. It turns out to be convenient to assume that all of our representations are equipped with an inner product, meaning that they are representations via the orthogonal group $O(n)$ (rather than the larger group $Gl_n(\mathbb{R})$).

Definition 2.2.8. A G-equivariant spectrum will mean a sequence $\{E_V\}$ of based G-spaces, one for each finite-dimensional orthogonal representation V, together with G-equivariant structure maps $\sigma_{V,W} \colon \Sigma^{W-V} E_V \longrightarrow E_W$ for all $V \subset W$, where $W - V$ denotes the orthogonal complement of V in W..

We will write $\mathbf{Sp}^G$ for the category of G-equivariant spectra. We then have

$$\boxed{\mathbf{Ho}(\mathbf{Sp}^G) \simeq \mathbf{Ho}(G\mathbf{Top}_*)[(S^\rho)^{-1}]\,.}$$

In the literature, these G-spectra are sometimes referred to as **"genuine" G-spectra**. We also have suspension spectra in this context:

Example 2.2.9. For any based G-space X, the (genuine) **equivariant suspension spectrum** of X, denoted $\Sigma^\infty_G X$, is the sequence $\{E_V = \Sigma^V X\}$, together with the identifcations $\Sigma^{W-V} E_V = \Sigma^{W-V}\Sigma^V X \cong \Sigma^W X = E_W$ for the maps $\sigma_{V,W}$. Note that here G is acting both on the suspension coordinates, and on X.

This defines functors

$$\Sigma^\infty_G \colon G\mathbf{Top}_* \longrightarrow \mathbf{Sp}^G \qquad \text{and} \qquad \Sigma^\infty_G \colon \mathbf{Ho}(G\mathbf{Top}_*) \longrightarrow \mathbf{Ho}(\mathbf{Sp}^G).$$

We write $\mathbf{S}^V_G$ for $\Sigma^\infty_G S^V$, and we can make sense of $\mathbf{S}^V_G$ for V any virtual representation in $RO(G)$. As a result, if we define a cohomology theory h^* from a G-spectrum E by the formula

$$h^n(X) = [\mathbf{S}^{-n} \wedge \Sigma^\infty_G X, E]_{\mathbf{Ho}(\mathbf{Sp}^G)},$$

then it is possible to extend this to a theory graded on the representation ring $RO(G)$ by the formula

$$h^V(X) = [\mathbf{S}^{-V} \wedge \Sigma^\infty_G X, E]_{\mathbf{Ho}(\mathbf{Sp}^G)}.$$

This is especially nice in the cases of $G = C_2$ or $G = C_3$ since the representation rings are rank two free abelian groups, meaning that we can display the groups of an $RO(G)$-graded theory in a grid. For a larger group, as the representation ring gets larger, it becomes impractical to display $RO(G)$-graded groups (much less to compute them). Nevertheless, the $RO(G)$-grading gives the possibility for a Poincaré duality theorem. For this, we first need a notion of orientability. And it turns out we need to go to $RO(G)$-grading to find orientations for G-bundles.

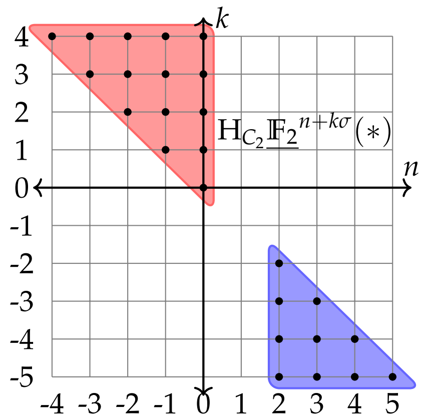
Definition 2.2.10. Let $\underline{N}$ be a ring Mackey functor, meaning that each $\underline{N}(G/H)$ is a commutative ring, each restriction map is a ring homomorphism, and each transfer map is a module map. Let X be a compact G-space, let ξ be an n-plane bundle on X, and let $T\xi$ be the Thom space of ξ, which in this case is the one-point compactification of the total space of ξ. Then an orientation of ξ will mean a class $\mu \in \widetilde{\mathrm{H}}^\alpha(T\xi; \underline{N})$ for some $\alpha \in RO(G)$ such that the restriction of μ along any inclusion $G/H \hookrightarrow X$ gives a generator of the free $\underline{N}(H)$-module $\widetilde{\mathrm{H}}^\alpha(Ti^*\xi; \underline{N})$.

Theorem 2.2.11 (Equivariant Poincaré Duality)**.** *Let $\underline{N}$ be a ring Mackey functor. Let M be a compact G-manifold with an orientation $\mu \in \mathrm{H}^\alpha(T\tau; \underline{N})$ of the tangent bundle τ. Then there is an isomorphism*

$$\mathrm{H}^\beta(M; \underline{N}) \cong \mathrm{H}_{\alpha-\beta}(M; \underline{N}).$$

Wed, Nov. 18

Last time, we said that cohomology theories arising from G-spectra (the ones indexed on all representations, not just S^1) can be thought of as $RO(G)$-graded rather than just $\mathbb{Z}$-graded. This turns out to be a useful point of view from the perspective of calculations.



Example 2.2.12. We of course first look at the case of $G = C_2$. Recall that $RO(C_2) \cong \mathbb{Z} \oplus \mathbb{Z}\{\sigma\}$. So a C_2-equivariant cohomology will have bigraded coefficient groups. We will talk through the calculation of $H_{C_2}\underline{\mathbb{F}_2}^{n+k\sigma}(*)$, the ordinary cohomology of a point with coefficients in the constant Mackey functor $\underline{\mathbb{F}_2}$. We display this in a grid to the right. Each dot in the picture represents a copy of $\mathbb{F}_2$.

We start by calculating the blue region. For $k > 0$, we wish to calculate the groups $\mathrm{H}^{n-k\sigma}(*)$. Using the suspension isomorphism, we get

$$\mathrm{H}\underline{\mathbb{F}_2}^{n-k\sigma}(*) \cong \widetilde{\mathrm{H}\underline{\mathbb{F}_2}}^{n-k\sigma}(S^0) \cong \widetilde{\mathrm{H}\underline{\mathbb{F}_2}}^{n}(S^{k\sigma}) = \widetilde{\mathrm{H}}^n_{C_2}(S^{k\sigma};\underline{\mathbb{F}_2}).$$

But now that we have translated this into a question of calculating $\mathbb{Z}$-graded groups, we are free to use Proposition 2.1.65, which gives us that

$$\widetilde{\mathrm{H}}^n_{C_2}(S^{k\sigma};\underline{\mathbb{F}_2}) \cong \widetilde{\mathrm{H}}^n(S^{k\sigma}/C_2;\mathbb{F}_2).$$

On the worksheet this week, I will ask you to verify that $S^{k\sigma}/C_2$ is homotopy equivalent to $\Sigma\mathbb{RP}^{k-1}$. Now the reduced cohomology groups of $\mathbb{RP}^{k-1}$ are $\mathbb{F}_2$ in dimensions $1, \dots, k-1$. It follows that $\Sigma\mathbb{RP}^{k-1}$ has reduced cohomology given by $\mathbb{F}_2$ in dimensions $2, \dots, k$. We conclude that, for $k > 0$, we have

$$\mathrm{H}\underline{\mathbb{F}_2}^{n-k\sigma}(*) \cong \widetilde{\mathrm{H}}^n(\Sigma\mathbb{RP}^{k-1};\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & 2 \le n \le k \\ 0 & \text{else.} \end{cases}$$

This gives us all of the dots in the blue cone.

Now we turn to the groups $\mathrm{H}\underline{\mathbb{F}_2}^{n+k\sigma}(*)$ with $k \geq 0$. Using duality, we get

$$\mathrm{H}\underline{\mathbb{F}_2}^{n+k\sigma}(*) \cong \widetilde{\mathrm{H}\underline{\mathbb{F}_2}}^{n+k\sigma}(S^0) \cong \widetilde{\mathrm{H}\underline{\mathbb{F}_2}}_{-n-k\sigma}(S^0) \cong \widetilde{\mathrm{H}\underline{\mathbb{F}_2}}_{-n}(S^{k\sigma}) = \widetilde{\mathrm{H}}_{-n}(S^{k\sigma};\underline{\mathbb{F}_2}),$$

where the last isomorphism is the suspension isomorphism. Now it is tempting to use part (1) of Corollary 2.1.76 to calculate these homology groups, but there is an unfortunate clash of notation here. In the homology groups here, $\underline{\mathbb{F}_2}$ stands for the constant Mackey functor at $\mathbb{F}_2$. Recall that for homology, we only use the $\mathcal{O}_G$-module coming from the transfer (and conjugation) maps of the Mackey functor. The transfer maps of $\underline{\mathbb{F}_2}$ are zero. On the other hand, in part (1) of Corollary 2.1.76, the $\underline{\mathbb{Z}}$ there denoted the constant $\mathcal{O}_G$-module, whose transfer map is the identity.

Now the $\mathcal{O}_G$-module $\underline{\mathbb{F}_2}$ coming from the constant Mackey functor, whose transfer map is zero, splits as a direct sum of $\mathcal{O}_G$-modules $\underline{\mathbb{F}_2} \cong \mathrm{Inf}(\mathbb{F}_2) \oplus \underline{f}$, where $\underline{f}$ is the $\mathcal{O}_G$-module that showed up in the proof of Theorem 2.1.81. This $\mathcal{O}_G$-module is just $\mathbb{F}_2$ at the bottom and zero at the top (opposite to $\mathrm{Inf}(\mathbb{F}_2)$). It follows that

$$\widetilde{\mathrm{H}}_{-n}(S^{k\sigma};\underline{\mathbb{F}_2}) \cong \widetilde{\mathrm{H}}_{-n}(S^{k\sigma};\mathrm{Inf}(\mathbb{F}_2)) \oplus \widetilde{\mathrm{H}}_{-n}(S^{k\sigma};\underline{f}).$$

By Corollary 2.1.76 and the homology analogue of Proposition 2.1.70, and using that $(S^{k\sigma})^{C_2} \cong S^0$, this can be rewritten as

$$\widetilde{\mathrm{H}}_{-n}(S^{k\sigma};\underline{\mathbb{F}_2}) \cong \widetilde{\mathrm{H}}_{-n}(S^0;\mathbb{F}_2) \oplus \widetilde{\mathrm{H}}_{-n}(((S^{k\sigma})/S^0)/C^2;\mathbb{F}_2). \tag{2.2.13}$$

The first term gives us a copy of $\mathbb{F}_2$ exactly when $n = 0$ (this gives the dots on the positive y-axis in the figure), while the second term is a bit of a disaster. But we can use the cofiber sequence

$$S(k\sigma)_+ \hookrightarrow D(k\sigma)_+ \longrightarrow S^{k\sigma},$$

where $S(V)$ and $D(V)$ denote the unit sphere and unit disk, respectively, inside an orthogonal representation. Note that actually $S(k\sigma) \cong S_a^{k-1}$, the anitpodal sphere of dimension $k-1$. Thus, using that $D(V)$ is equivariantly contractible, upon passing to the quotient by the C_2-action, we get a cofiber sequence

$$\mathbb{RP}^{k-1}_+ \longrightarrow S^0 \longrightarrow S^{k\sigma}/C_2 \longrightarrow \Sigma\mathbb{RP}^{k-1}_+.$$

Thus we can rewrite (2.2.13) as

$$\begin{aligned}\widetilde{\mathrm{H}}_{-n}(S^{k\sigma};\underline{\mathbb{F}_2}) &\cong \widetilde{\mathrm{H}}_{-n}(S^0;\mathbb{F}_2) \oplus \widetilde{\mathrm{H}}_{-n}(\Sigma\mathbb{RP}^{k-1}_+;\mathbb{F}_2)\\ &\cong \widetilde{\mathrm{H}}_{-n}(S^0;\mathbb{F}_2) \oplus \widetilde{\mathrm{H}}_{-n-1}(\mathbb{RP}^{k-1}_+;\mathbb{F}_2).\end{aligned}$$

The first term contributes an $\mathbb{F}_2$ when $n = 0$, and the second term contributes an $\mathbb{F}_2$ when $n-1 \in [0, k-1]$, or in other words when $n \in [1, k]$. This gives us all of the dots in the red cone.

Watch the video (in canvas): An alternative point of view on the C_2-equivariant cohomology of a point

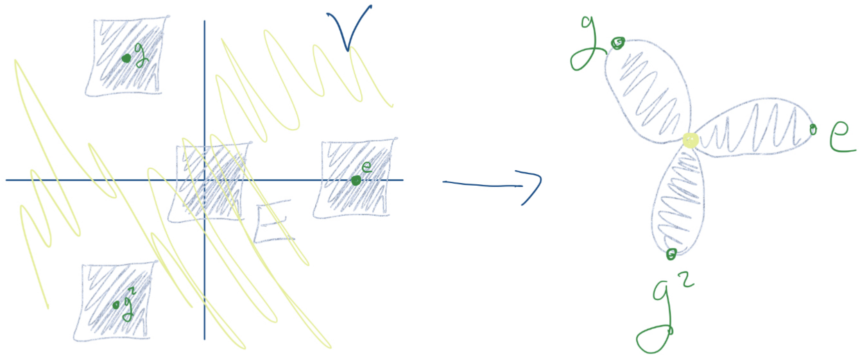
2.2.4. *Transfers and Mackey functors.* Recall our discussion of transfer maps in Section 2.1.5. If $p\colon E \longrightarrow B$ is a covering map of spaces, there exists a wrong-way transfer map on homology and cohomology. One way to say this is that there exists a wrong-way transfer map $\Sigma^\infty B_+ \xrightarrow{p^!} \Sigma^\infty E_+$ on the associated suspension spectra such that the composition

$$\Sigma^\infty B_+ \xrightarrow{p^!} \Sigma^\infty E_+ \xrightarrow{\Sigma^\infty p} \Sigma^\infty B_+$$

is multiplication by the degree of the cover. We want to see that something like this happens equivariantly as well.

For simplicity, we work with the elementary example of $p\colon G/H \longrightarrow *$. Note that, were we ignoring the equivariance, the transfer map we would be looking for would be of the form $\mathbf{S}^0 = \Sigma^\infty *_+ \longrightarrow \Sigma^\infty n_+$, where $n = |G/H|$. But remember that the stable homotopy category is an additive category, and $\Sigma^\infty n_+$ is $\bigvee_n \mathbf{S}^0$. But in any additive category, a finite coproduct is the same as a finite product, usually written as a direct sum. In other words, the desired transfer map is of the form $\mathbf{S}^0 \longrightarrow \bigoplus_n \mathbf{S}^0$, and we would just take the diagonal map.

Back to the equivariant situation, we want to start by embedding our G-orbit G/H into a representation V. For example, we can always take V to be the permutation representation $\mathbb{R}[G/H]$ with G/H embedding inside $V = \mathbb{R}[G/H]$ as the basis elements. Now let $E \subset V$ be a disk of small radius ϵ around the origin, so that $E \cong V$. For any coset representative gH, which we view as a nonzero vector of V, we then get the translate $gH + E \subset V$. We then define the **Thom collapse** map $V \longrightarrow G/H_+ \wedge V$ as displayed below: that is, every element outside of the translates $gH + E$ gets collapse to the basepoint. In the picture, we have $G = C_3$ and $V = \lambda_3$.



This extends to a map $S^V \longrightarrow G/H_+ \wedge S^V$ on the one-point compactification. In the homotopy category $\mathbf{Ho}(\mathbf{Sp}^G)$, we can desuspend this map by S^V to obtain a map $\mathbf{S}^0 \longrightarrow \Sigma^\infty G/H_+$ as desired, and this is the transfer map. This lends credence to the following statement.

Proposition 2.2.14. *Let $E \in \mathbf{Sp}^G$ be a G-spectrum, and define*

$$\underline{M}(G/H) = [\Sigma_G^\infty (G/H)_+, E]_{\mathbf{Ho}(\mathbf{Sp}^G)}.$$

Then $\underline{M}$ inherits the structure of a G-Mackey functor.

In fact, more is true:

Theorem 2.2.15. *Let M be an $\mathcal{O}_G$-module. Then the ($\mathbb{Z}$-graded) equivariant homology theory $\mathrm{H}_*(-;M)$ extends to an $RO(G)$-graded theory if and only if M extends to a Mackey functor.*

The condition in the theorem is a real condition. For instance, the $\mathcal{O}_G$-module $\underline{F} = \begin{matrix} 0 \\ \downarrow \\ \mathbb{Z} \end{matrix}$ introduced just after Proposition 2.1.69 cannot extend to a Mackey functor. This is because, since the C_2 action on $\underline{F}(e)$ is trivial, the double-coset formula will tell you that applying the transfer and then the restriction to an element in $\underline{F}(e)$ should simply multiply that element by 2. Since $\underline{F}(C_2) = 0$, this is impossible.

Mon, Nov. 23

2.2.5. *π_0 of the equivariant sphere spectrum.* Classically, Freudenthal gives that $\pi_0(\mathbf{S}^0) \cong \mathbb{Z}$. There is an equivariant version of the Freudenthal theorem, though it is a bit more complicated to state. We give a simplified form, but see [M, Theorem IX.1.4] for a more complete version.

Theorem 2.2.16 (Equivariant Freudenthal Suspension Theorem)**.** *Let X be a finite-dimensional G-CW complex and Y be a G-space. Then there exists a G-representation W such that the V-suspension isomorphism*

$$[S^W \wedge X, S^W \wedge Y]^G \longrightarrow [S^{V\oplus W} \wedge X, S^{V\oplus W} \wedge Y]^G$$

is an isomorphism for all representations V.

For example, in the case $G = C_p$ the stabilization maps

$$[S^{k\rho}, S^{k\rho}]^G \longrightarrow [S^{(k+1)\rho}, S^{(k+1)\rho}]^G$$

are isomorphisms if $k \geq 1$. These stable values are the equivariant stable homotopy group $\pi_0(\mathbf{S}^0_G)$, which has the following nice description.

Theorem 2.2.17. *$\pi_0(\mathbf{S}^0_G) \cong A(G)$ and $\underline{\pi}_0(\mathbf{S}^0_G) \cong \underline{A}(G)$.*

Proof. Recall that in Section 2.2.4, we defined a transfer map in $\mathbf{Sp}^G$ associated to any finite G-covering. In particular, we have $p^H_!\colon \mathbf{S}^0_G \longrightarrow \Sigma^\infty_G G/H_+$ associated to the cover $p^H\colon G/H \longrightarrow *$. We define

$$\chi\colon A(G) \longrightarrow \pi_0(\mathbf{S}^0_G)$$

by $\chi(G/H) = \mathbf{S}^0_G \xrightarrow{p^H_!} \Sigma^\infty_G G/H_+ \xrightarrow{p^H} \mathbf{S}^0_G$. We have denoted this map by χ because it is in fact an Euler characteristic. We claim that this map is an isomorphism. Now define, for each $K \leq G$, a homomorphism

$$d_K\colon \pi_0(\mathbf{S}^0_G) \longrightarrow \mathbb{Z}$$

by first representing a stable map as a map $f\colon S^V \longrightarrow S^V$ between representation spheres and then taking the degree of the induced map $f^K\colon S^{V^K} \longrightarrow S^{V^K}$ on K-fixed spheres. Now we claim that the composite

$$A(G) \xrightarrow{\chi} \pi_0(\mathbf{S}^0_G) \xrightarrow{d_K} \mathbb{Z}$$

sends G/H to the cardinality $|(G/H^K|$ of the K-fixed points. The idea is that applying K-fixed points to the composition

$$\mathbf{S}^0_G \xrightarrow{p^H_!} \Sigma^\infty_G G/H_+ \xrightarrow{p^H} \mathbf{S}^o_G$$

gives the composition

$$\mathbf{S}^0 \xrightarrow{q_!} \Sigma^\infty (G/H)^K_+ \xrightarrow{q} \mathbf{S}^0,$$

where q is the map $(G/H)^K \longrightarrow *$. Now we use the fact that in the nonequivariant stable homotopy category, this composition gives the Euler characteristic of $(G/H)^K$. Since this is a finite set, the Euler characteristic is just the cardinality.

If we assemble these maps d_K as K varies over the conjugacy classes of subgroups of G, we obtain

$$A(G) \xrightarrow{\chi} \pi_0(\mathbf{S}^0_G) \xrightarrow{(d_K)} \prod_{[K]} \mathbb{Z}.$$

This composition is sometimes called the **ghost map** or the **mark homomorphism**, and the target the **ghost ring** or **ring of marks**. We claim it is injective, which implies that χ is also injective. To see this, choose an ordering of the conjugacy classes of subgroups such that $(H_1) < (H_2)$ whenever

H_1 is subconjugate to H_2. As in the the discussion just after Definition 2.1.50, the K-fixed points of G/H are nonempty if and only if K is subconjugate to H. This implies that if we represent the mark homomorphism as a matrix (this matrix is known as the **table of marks**), then this matrix is upper triangular, with diagonal entries given by the orders of the Weyl groups. It follows that the mark homomorphism is injective.

It remains to show that χ is surjective. Thus suppose given an element of $\pi_0(\mathbf{S}^0_G)$, represented by a G-equivariant map $f\colon S^V \longrightarrow S^V$ for some representation V. We make three claims (without proof), which together show that χ is surjective:

Claim 1: Writing $n_K = d_K(f)$, the tuple (n_K) satisfies the congruence

$$\sum_K |N_G H\colon N_G H \cap N_G K| \cdot \{\#\,\text{generators of } K/H\} \cdot n_K \equiv 0 \pmod{|W_G H|} \tag{2.2.18}$$

for every H, where the sum runs over conjugacy classes of subgroups such that H is normal in K and K/H is cyclic.

Claim 2: Any tuple (n_K) of integers satisfying the congruence (2.2.18) for every H is in the image of the mark homomorphism.

This is not too difficult. The key is that the congruence (2.2.18) can equivalently be written as

$$\sum_K |\{x \in W_G(H) \text{ such that } \langle x, H\rangle = K\}| \cdot n_K \equiv 0 \pmod{|W_G H|},$$

where now the sum runs over all conjugacy classes of subgroups.

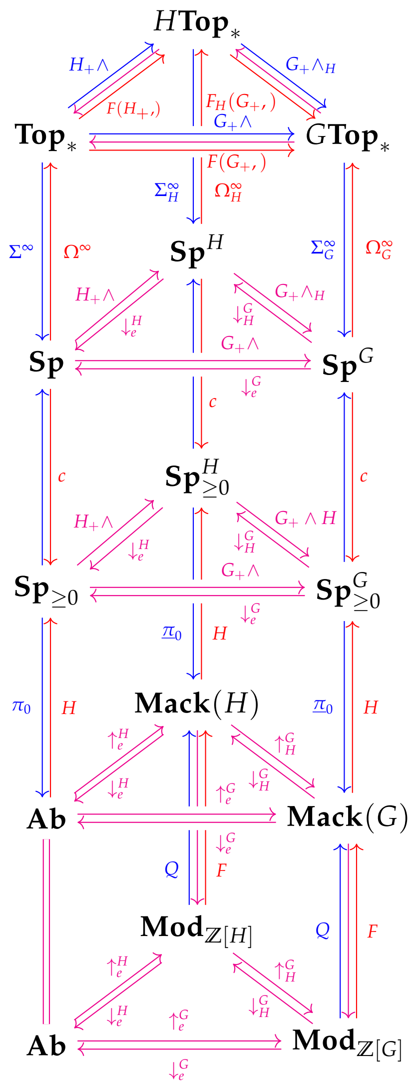
Claim 3: The map $\pi_0(\mathbf{S}^0_G) \xrightarrow{(d_K)} \prod_{[K]} \mathbb{Z}$ is injective.

This can be shown by obstruction theory.

∎

2.3. **The big diagrams.**

We display a pair of diagrams here relating many of the categories we have discussed: G-modules, Mackey functors, G-spaces, and G-spectra.



Left adjoints are displayed in blue, right adjoints in red, and functors that are both are displayed in magenta. The diagram commutes, in the sense that every square of left adjoints commutes (and similarly with right adjoints). We use the shorthand notation

$$F(H_+, X) = \mathrm{Map}(H_+, X), \quad F_H(G_+, X) = \mathrm{Map}_H(G_+, X)$$

(think function space). Also, in each case (for both spaces and spectra), we mean the respective *homotopy categories*.

There are several important points here:

(1) In all contexts except for the top one (for spaces), we have a "coinduction is induction" result, like Proposition 1.1.46. For spaces, we know that induction and coinduction cannot agree, as you saw on Worksheet 10.

(2) In the world of G-spectra, the statement that coinduction agrees with induction is known as the "Wirthmuller isomorphism". By the shearing isomorphism, for any G-spectrum, it gives an isomorphism

$$G/H_+ \wedge X \simeq F(G/H_+, X).$$

In the case of $X = \mathbf{S}^0_G$, this tells us that the suspension spectrum $\Sigma^\infty_G G/H_+$ is **self-dual** in $\mathbf{Ho}(\mathbf{Sp}^G)$.

And a fixed points diagram:

$$
\begin{tikzcd}
& H\mathbf{Top}_* \arrow[dl, shift left, "(-)^H"] \arrow[dr, shift left, "G_+\wedge_H"] \arrow[dd, shift right, "\Sigma^\infty_H"'] & \\
\mathbf{Top}_* \arrow[ur, shift left, "\mathrm{triv}"] \arrow[rr, shift left, "G/H_+\wedge"] \arrow[dd, shift right, "\Sigma^\infty"'] & & G\mathbf{Top}_* \arrow[ul, shift left, "\downarrow^G_H"] \arrow[ll, shift left, "(-)^H"] \arrow[dd, shift right, "\Sigma^\infty_G"'] \\
& \mathbf{Sp}^H \arrow[uu, shift right, "\Omega^\infty_H"'] \arrow[dl, shift left, "(-)^H"] \arrow[dr, shift left, "G_+\wedge_H"] \arrow[dd, shift right, "c"'] & \\
\mathbf{Sp} \arrow[uu, shift right, "\Omega^\infty"'] \arrow[ur, shift left, "\mathrm{triv}"] \arrow[rr, shift left, "G/H_+\wedge"] \arrow[dd, shift right] & & \mathbf{Sp}^G \arrow[uu, shift right, "\Omega^\infty_G"'] \arrow[ul, shift left, "\downarrow^G_H"] \arrow[ll, shift left, "(-)^H"] \arrow[dd, shift right] \\
& \mathbf{Sp}^H_{\geq 0} \arrow[uu, shift right] \arrow[dl, shift left, "(-)^H"] \arrow[dr, shift left, "G_+\wedge_H"] \arrow[dd, shift right, "\pi_0"'] & \\
\mathbf{Sp}_{\geq 0} \arrow[uu, shift right, "c"'] \arrow[ur, shift left, "\mathrm{triv}"] \arrow[rr, shift left, "G/H_+\wedge"] \arrow[dd, shift right, "\pi_0"'] & & \mathbf{Sp}^G_{\geq 0} \arrow[uu, shift right, "c"'] \arrow[ul, shift left, "\downarrow^G_H"] \arrow[ll, shift left, "(-)^H"] \arrow[dd, shift right, "\pi_0"'] \\
& \mathbf{Mack}(H) \arrow[uu, shift right, "H"'] \arrow[dl, shift left, "\mathrm{ev}_H"] \arrow[dr, shift left, "\uparrow^G_H"] & \\
\mathbf{Ab} \arrow[uu, shift right, "H"'] \arrow[ur, shift left, "A_H\otimes"] \arrow[rr, shift left, "\uparrow^G_H A_H\otimes"] & & \mathbf{Mack}(G) \arrow[uu, shift right, "H"'] \arrow[ul, shift left, "\downarrow^G_H"] \arrow[ll, shift left, "\mathrm{ev}_H"]
\end{tikzcd}
$$

As on the previous page, left adjoints are displayed in blue, right adjoints in red, and functors that are both are displayed in magenta. The diagram commutes, in the sense that every square of left adjoints commutes (and similarly with right adjoints).
Again, we cannot draw conclusions about composing left adjoints with right adjoints. For example, we might wonder about the composition

$$G\mathbf{Top}_* \xrightarrow{\Sigma^\infty_G} \mathbf{Sp}^G \xrightarrow{(-)^G} \mathbf{Sp},$$

but cannot get our hopes too high since we are composing a left and a right adjoint. In fact, this composition is understood, but the answer is more complicated than you might have hoped.

Theorem 2.3.1 (tom Dieck splitting)**.** *Let X be a based G-space. Then*

$$(\Sigma^\infty_G X)^G \simeq \bigvee_{(H)} \Sigma^\infty (X^H)_{hW_G(H)},$$

where the subcript $hW_G(H)$ denotes the homotopy orbits (a.k.a. Borel cosntruction) of the Weyl group on the H-fixed points.

For example, taking $G = C_2$, we obtain

$$(\mathbf{S}^0_{C_2})^{C_2} = (\Sigma^\infty_{C_2} S^0)^{C_2} \simeq \mathbf{S}^0 \vee \Sigma^\infty \mathbb{RP}^\infty_+.$$

References

[B] A. Blumberg, *Lecture Notes on Equivariant Stable Homotopy Theory*. Available at `https://web.ma.utexas.edu/users/a.debray/lecture_notes/m392c_EHT_notes.pdf`.

[M] J. P. May, *Equivariant homotopy and cohomology theory*, CBMS Regional Conference Series in Mathematics, vol. 91, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996. Available at `https://web.math.rochester.edu/people/faculty/doug/papers.html`. MR1413302

[TW] J. Thevenaz and P Webb, *The Structure of Mackey Functors*. Available at `https://web.math.rochester.edu/people/faculty/doug/otherpapers/thevenaz-webb.pdf`.

[W] P. Webb, *A Guide to Mackey Functors*. Available at `https://web.math.rochester.edu/people/faculty/doug/otherpapers/WebbMF.pdf`.