The subalgebras $\mathcal{A}(n)$ & nilpotence in $\mathcal{A}^*$.

Goal: • Every elt in $\mathcal{A}^{>0}$ is nilpotent

• More generally, any finite collection $Sq^{i_1}, \ldots, Sq^{i_n}$ generates a finite subalgebra of $\mathcal{A}^*$.

Will show $\{Sq^1, Sq^2, Sq^4, \ldots, Sq^{2^n}\}$ generates a finite subalgebra.

Recall Milnor basis for $\mathcal{A}^*$. Start w/ Serre-Cartan basis of admissibles for $\mathcal{A}^*$. Define classes $\xi_i \in \mathcal{A}_{2^i-1}$

wrt adm basis

dual to $Sq^{2^{i-1}} Sq^{2^{i-2}} \cdots Sq^2 Sq^1$.

Thm (Milnor) $\mathcal{A}_* \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots]$

$$\Delta(\xi_n) = \sum_i \xi_{n-i}^{2^i} \otimes \xi_i$$

Now define $Sq^{r_1, r_2, \ldots, r_n} \in \mathcal{A}^*$ of degree $r_1 + 3r_2 + 7r_3 + \cdots + (2^n-1)r_n$

wrt monomial basis

dual to $\xi_1^{r_1} \xi_2^{r_2} \cdots \xi_n^{r_n}$

This is the Milnor basis for $\mathcal{A}_*$.

<u>Be careful!</u> ξ_i dual to $Sq^{2^{i-1}} \cdots Sq^1$ and $Sq^{0,0,\ldots,1}$ ($\nearrow$ position i) dual to ξ_i, but this does <u>not</u> imply that $Sq^{0,0,\ldots,1} = Sq^{2^{i-1}} \cdots Sq^1$.

<u>e.g.</u> What is $Sq^{0,1}$ in $\mathcal{A}^3$? Adm. basis is $\{Sq^3, Sq^2Sq^1\}$.

Milnor basis for $\mathcal{A}_3$ is $\{\xi_1^3, \xi_2\}$.

Write $\langle\ ,\ \rangle : \mathcal{A}_n \otimes \mathcal{A}^n \longrightarrow \mathbb{F}_2$ (evaluation).

By defn of $Sq^{0,1}$, have $\langle \xi_1^3, Sq^{0,1}\rangle = 0$

$\langle \xi_2, Sq^{0,1} \rangle = 1.$

Which lin. comb. of Sq^3 and Sq^2Sq^1 satisfies this?

Calculate:

	ξ_1^3	ξ_2
Sq^3	1 (2)	0
Sq^2Sq^1	1 (3)	1
$Sq^3 + Sq^2Sq^1$	0 (4)	1

(1)

Recall $\langle x \cdot y, z \rangle$

$= \langle x \otimes y, \Delta(z) \rangle$

$= \sum_i \langle x, z_i' \rangle \langle y, z_i'' \rangle$

where $\Delta(z) = \sum_i z_i' \otimes z_i''$.

So $\langle \xi_1^3, Sq^3 \rangle = \langle \xi_1 \otimes \xi_1^2, Sq^3 \otimes 1 + Sq^2 \otimes Sq^1 + Sq^1 \otimes Sq^2 + 1 \otimes Sq^3 \rangle$

$= \underset{1=}{\langle \xi_1, Sq^1 \rangle} \cdot \langle \xi_1^2, Sq^2 \rangle$

$= \langle \xi_1 \otimes \xi_1, Sq^2 \otimes 1 + Sq^1 \otimes Sq^1 + 1 \otimes Sq^2 \rangle$

$= \langle \xi_1, Sq^1 \rangle \cdot \langle \xi_1, Sq^1 \rangle = 1.$

$$\langle \xi_1^3, Sq^2Sq^1\rangle = \left\langle \xi_1 \otimes \xi_1^2, \left(Sq^2\otimes 1 + Sq^1\otimes Sq^1 + 1\otimes Sq^2\right) \cdot \left(Sq^1\otimes 1 + \cancel{Sq}\, 1\otimes Sq^1\right)\right\rangle$$

$$= \langle \xi_1, Sq^1\rangle \langle \xi_1^2, \cancel{Sq^1Sq^1} + Sq^2\rangle$$

$$= 1$$

<u>Conclusion:</u> $Sq^{0,1} = Sq^3 + Sq^2Sq^1$.

<u>Remark</u> Easy to see that $Sq^{n,0,0,0} = Sq^n$.

Consider quotient

$$\mathcal{A}_* \twoheadrightarrow \mathcal{A}_*/\xi_j^{2^{n+2-j}} \quad \text{all } j$$

$$\mathcal{A}(n)_* = \mathcal{A}_*/\xi_j^{2^{n+2-j}} \cong \frac{\mathbb{F}_2[\xi_1, \ldots, \xi_{n+1}]}{(\xi_1^{2^{n+1}}, \xi_2^{2^n}, \ldots, \xi_{n+1}^2)}$$

<u>Note:</u> $\dim_{\mathbb{F}_2} \mathcal{A}(n)_* = 2^{n+1}\cdot 2^n \cdot \ldots \cdot 2 = 2^{n+1+n+\cdots+2} = 2^{\binom{n+2}{2}}$.

Top degree is $2^{n+1} - 1 + (4-1)(2^n - 1) + \cdots + (2^{n+1} - 1)$

$$= (n-1)\, 2^{n+2} + n + 5 \qquad (\text{for } n \geq 1)$$

Also, coproduct formula for $\Delta(\xi_k^{2^{n+2-k}})$ shows this is a map of <u>Hopf</u> <u>algebras</u>.

Dualizing gives an inclusion

$$\mathcal{A}(n)^* \hookrightarrow \mathcal{A}^* \quad \text{of a finite Hopf subalgebra.}$$

Clear that $Sq^1, Sq^2, \dots, Sq^{2^n} \in \mathcal{A}(n)^*$

since these are dual to $\xi_1, \xi_1^2, \dots, \xi_1^{2^n} \in \mathcal{A}(n)_*$.

$\Rightarrow Sq^1, \dots, Sq^{2^n}$ generate a subalgebra of $\mathcal{A}(n)^*$

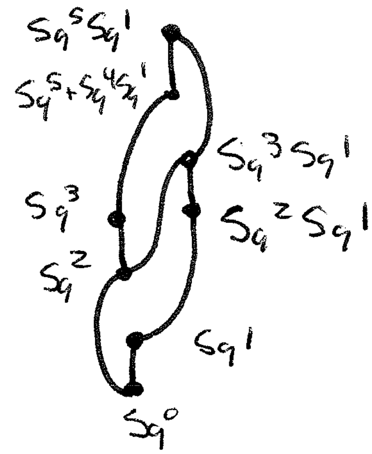
so this must be finite.

(In fact, they generate $\mathcal{A}(n)^*$. See Milnor, §8).

e.g. • $\mathcal{A}(0) = \mathbb{F}_2[Sq^1]/(Sq^1)^2$

• $\mathcal{A}(1)$ gen. by Sq^1, Sq^2.

Draw picture



In dim ≤ 2, only one class.

In dim 3, adm basis

is Sq^2Sq^1, Sq^3 $(= Sq^1Sq^2)$

In dim 4, adm basis is

$\cancel{Sq^4}$, Sq^3Sq^1

By Adem, $Sq^2Sq^2 = \binom{1}{2} Sq^4 + \binom{0}{0} Sq^3Sq^1 = Sq^3Sq^1$

Dim 5: $Sq^2Sq^2Sq^1 = Sq^3Sq^1Sq^1 = 0$, $Sq^2Sq^3 = \binom{2}{2} Sq^5 + \binom{*}{0} Sq^4Sq^1 = Sq^5 + Sq^4Sq^1$

Dim 6: $Sq^1Sq^2Sq^3 = Sq^5Sq^1$,

$Sq^2Sq^3Sq^1 = Sq^5Sq^1 + \cancel{Sq^4Sq^1}Sq^1$

Already saw $Sq^2 Sq^2 Sq^1 = 0$, so no class in dim 7.
or dim 8.

So this picture represents $\mathcal{A}(1)^*$, of total dimension 8

w/ top class in dimension 6.

See website for pictures of $\mathcal{A}(2)$.

<u>Remark</u>: In general, no formula for smallest k such

that $(Sq^n)^k = 0$. Known in <u>some</u> cases:

- $\left(Sq^{2^m}\right)^{2m+2} = 0$, $\left(Sq^{2m}\right)^{2m+1} \neq 0$
- $\left(Sq^{2^m-1}\right)^{m+1} = 0$, $\left(Sq^{2^m-1}\right)^{m} \neq 0$.
- Sq^{2^m-2} <u>not</u> known