KAN'S Ex^∞ FUNCTOR

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1. References

(1) P. G. Goerss and J. F. Jardine, *Simplicial Homotopy Theory*, Progress in Mathematics **174**, Birkhäuser, 1999.
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2. Simplicial Sets

We begin with a quick review of simplicial sets and Kan complexes.

The category $\mathbf{\Delta}$ has objects the totally ordered sets $\mathbf{n} = \{0, 1, \ldots, n\}$ for $n \geq 0$. A morphism $\mathbf{m} \to \mathbf{n}$ is an order-preserving map. For each $0 \leq i \leq n$, there are maps

$$d^i : \mathbf{n-1} \to \mathbf{n} \quad \text{(cofaces)}$$

and

$$s^i : \mathbf{n+1} \to \mathbf{n} \quad \text{(codegeneracies)},$$

where d^i omits the element i and s^i is the surjection such that $s^i(i) = s^i(i+1) = i$.

Definition 1. A **simplicial set** is simply a contravariant functor $X : \mathbf{\Delta}^{op} \to Set$. We often denote $X(\mathbf{n})$ by X_n. A morphism of simplicial sets is simply a natural transformation. The category of simplicial sets will be denoted $sSet$.

Example 1. For each n, we have a simplicial set Δ^n given by

$$\Delta^n_m = \mathrm{Hom}_{sSet}(\mathbf{m}, \mathbf{n}).$$

The identity map $\mathbf{n} \to \mathbf{n}$ gives an element of Δ^n_n, which we denote ι_n.

Example 2. For each n, we have a subcomplex $\partial\Delta^n \subseteq \Delta^n$, which is the smallest simplicial set containing $d_i(\iota_n)$ for each $0 \leq i \leq n$.

Example 3. For each n, we have a subcomplex $\Lambda^n_k \subseteq \Delta^n$, which is the smallest simplicial set containing $d_i(\iota_n)$ for each $0 \leq i \leq n$, except for $i = k$.

Proposition 1. *For each simplicial set X, there is a canonical isomorphism*

$$\operatorname*{colim}_{\Delta^n \to X} \Delta^n \cong X,$$

where the colimit ranges over all maps from a standard n-simplex (for all n) to X.

Definition 2. The **geometric realization** functor $|-| : sSet \to Top$ is defined to be the colimit-preserving functor satisfying $|\Delta^n| = \Delta[n]$, where

$$\Delta[n] = \Big\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \sum_{i=0}^n x_i = 1\Big\}$$

is the geometric n-simplex.

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Definition 3. A map $f : X \to Y$ of simplicial sets is said to be a **weak equivalence** if $|f| : |X| \to |Y|$ induces an isomorphism

$$|f|_* : \pi_n(|X|, x) \cong \pi_n(|Y|, |f|(x))$$

for all $n \geq 0$ and all choices of basepoint $x \in |X|$.

Definition 4. A map $f : X \to Y$ is a **Kan fibration** if for every $n \geq 1$ and $0 \leq k \leq n$ there exists a lift in any diagram of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

A simplicial set is called a **Kan complex** if the map $X \to *$ is a fibration.

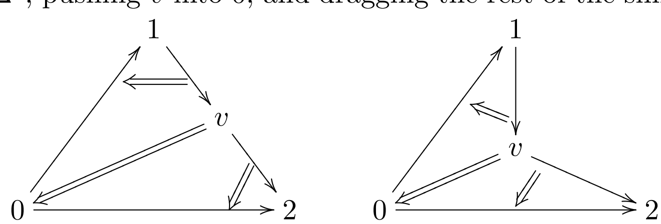
3. Fibrant replacement

The goal of the talk will be to introduce Kan's fibrant replacement functor Ex^∞. Thus for every simplicial set X there is a monomorphic weak equivalence $X \hookrightarrow \mathrm{Ex}^\infty(X)$ where $\mathrm{Ex}^\infty(X)$ is a Kan complex.

Suppose given a simplicial set X together with a lifting problem $\Lambda^n_k \to X$. Clearly when $n = 1$ there is no trouble in obtaining a lift, so let us start with $n = 2$ and $k = 0$. Thus we have a lifting problem

$$\begin{tikzcd} \Lambda^2_0 \arrow[r] \arrow[d, hook] & X. \\ \Delta^2 & \end{tikzcd}$$

To see how we might produce a lift, we pass to topology. The inclusion $|\Lambda^2_0| \hookrightarrow |\Delta^2|$ has a retraction. Such a retraction, for instance, is given by adding a vertex v at the midpoint of the edge $1 \to 2$ of Δ^2, pushing v into 0, and dragging the rest of the simplex along with it.



Thus we are beginning to see that we might be able to solve the Kan lifting problems by subdividing simplices. We now proceed to make this into a formal argument.

3.1. Subdivision

The nondegenerate m-simplices of the standard n-simplex Δ^n correspond to the subsets of $\{0, \ldots, n\}$ of size $m+1$. Thus, the nondegenerate simplices form a poset $\mathrm{nd}\Delta^n$, ordered by inclusion.

Definition 5. The subdivision of Δ^n is defined to be $\mathrm{sd}\,\Delta^n = \mathcal{N}\mathrm{nd}\Delta^n$, the nerve of the poset of nondegenerate simplices. For an arbitrary simplicial set X, we define

$$\mathrm{sd}\,X = \operatorname*{colim}_{\Delta^n \to X} \mathrm{sd}\,\Delta^n.$$

Example 4. The subdivision of Δ^1 is $0 \to (01) \leftarrow 1$.

Example 5. The subdivision of Δ^2 is

$$\begin{tikzcd}
 & & 1 \arrow[dl] \arrow[d] \arrow[dr] & & \\
 & 01 \arrow[r] & 012 & 12 \arrow[l] & \\
0 \arrow[ur] \arrow[urr] \arrow[rr] & & 02 \arrow[u] & & 2 \arrow[ul] \arrow[ull] \arrow[ll]
\end{tikzcd}$$

Definition 6. For $X \in sSet$, we define $\mathrm{Ex}(X)$ by

$$\mathrm{Ex}(X)_n = \mathrm{Hom}_{sSet}(\mathrm{sd}\,\Delta^n, X).$$

Note that since $Y_n \cong \mathrm{Hom}_{sSet}(\Delta^n, Y)$ for any simplicial set Y, the functor $\mathrm{Ex} : sSet \to sSet$ is right adjoint to sd.

Before defining the functor Ex^∞, we need one last ingredient.

Definition 7. The **last vertex map** $lv : \mathrm{sd}\,\Delta^n \to \Delta^n$ is the map induced by the map of posets $\mathrm{nd}\Delta^n \to \mathbf{n}$ given by

$$(i_0, \dots, i_m) \mapsto i_m.$$

This extends to give a map $lv : \mathrm{sd}\,X \to X$ for any simplicial set. Adjoint to lv is a map $j : X \to \mathrm{Ex}(X)$.

For any simplicial set X, we may consider the directed system

$$X \xrightarrow{j_X} \mathrm{Ex}(X) \xrightarrow{j_{\mathrm{Ex}(X)}} \mathrm{Ex}^2(X) \xrightarrow{j_{\mathrm{Ex}^2(X)}} \dots.$$

We denote the colimit by $\mathrm{Ex}^\infty(X)$, and this will be our fibrant replacement functor.

We will also need one other useful property of the subdivision functor. We claim that for any *function* $f : \mathbf{m} \to \mathbf{n}$ (so f is not necessarily order-preserving) we get a map $f_* : \mathrm{sd}\,\Delta^m \to \mathrm{sd}\,\Delta^n$ of simplicial sets. This holds since f still induces a map of posets $\mathrm{nd}\Delta^m \to \mathrm{nd}\Delta^n$.

4. Properties of Ex^∞

In this section we will state and prove important properties of Ex^∞. Note that from the construction, Ex^∞ is a functor.

Properties of Ex^∞:

(1) $\mathrm{Ex}^\infty(X)$ is a Kan complex for any simplicial set X
(2) $j_X : X \to \mathrm{Ex}^\infty(X)$ is an acyclic cofibration for any simplicial set X
(3) Ex^∞ preserves fibrations
(4) Ex^∞ preserves finite limites
(5) Ex^∞ preserves 0-simplices

Proof of the properties:
(1) It suffices to prove the following:

Lemma 1. *For any horn $\lambda : \Lambda^n_k \to \mathrm{Ex}(X)$, there exists an extension as in the diagram*

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\lambda"] \arrow[d, hook] & \mathrm{Ex}(X) \arrow[d, "j_{\mathrm{Ex}(X)}"] \\
\Delta^n \arrow[r, dashed] & \mathrm{Ex}^2(X).
\end{tikzcd}$$

Before proving the lemma, we show that it is *not* true that we can find extensions in diagrams of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "j"] \\ \Delta^n \arrow[r, dashed] & \mathrm{Ex}(X) \end{tikzcd}$$

Let us look in particular at the case $n = 2$, $k = 0$, $X = \Lambda^2_0$, with the horizontal map being the identity. Then the problem becomes equivalent to finding a lift in the diagram

$$\begin{tikzcd} \mathrm{sd}\,\Lambda^2_0 \arrow[r, "lv"] \arrow[d, hook] & \Lambda^2_0 \\ \mathrm{sd}\,\Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

Such a lift would necessarily send the barycenter to the vertex 2, so the edge from the vertex 1 to the barycenter would have nowhere to go in Λ^2_0.

Proof of Lemma 1. Note that $\lambda : \Lambda^n_k \to \mathrm{Ex}(X)$ factors as $\Lambda^n_k \xrightarrow{\eta} \mathrm{Ex}(\mathrm{sd}\,\Lambda^n_k) \xrightarrow{\mathrm{Ex}(\tilde{\lambda})} \mathrm{Ex}(X)$. Thus it suffices to obtain an extension as in the diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\eta"] \arrow[d, hook] & \mathrm{Ex}(\mathrm{sd}\,\Lambda^n_k) \arrow[r, "\mathrm{Ex}(\tilde{\lambda})"] \arrow[d, "j"] & \mathrm{Ex}(X) \arrow[d, "j"] \\ \Delta^n \arrow[r, dashed] & \mathrm{Ex}^2(\mathrm{sd}\,\Lambda^n_k) \arrow[r, "\mathrm{Ex}^2(\tilde{\lambda})"] & \mathrm{Ex}^2(X) \end{tikzcd}$$

But the composite $j \circ \eta : \Lambda^n_k \to \mathrm{Ex}^2(\mathrm{sd}\,\Lambda^n_k)$ agrees, by naturality of j, with

$$\Lambda^n_k \xrightarrow{j} \mathrm{Ex}\,\Lambda^n_k \xrightarrow{\mathrm{Ex}\,\eta} \mathrm{Ex}^2\,\mathrm{sd}\,\Lambda^n_k.$$

Thus by adjointness it suffices to find an extension in the diagram

$$\begin{tikzcd} \mathrm{sd}\,\Lambda^n_k \arrow[r, "lv"] \arrow[d, hook] & \Lambda^n_k \arrow[d, "\eta"] \\ \mathrm{sd}\,\Delta^n \arrow[r, dashed] & \mathrm{Ex}\,\mathrm{sd}\,\Lambda^n_k \end{tikzcd}$$

For each q-simplex $\sigma = (\sigma_0, \ldots, \sigma_q) \in \mathrm{sd}(\Delta^n)_q$ with $\sigma_i \in \Delta^n_{n_i}$, we define a *function* $f_\sigma : \mathbf{q} \to \mathbf{n}$ by

$$f_\sigma(i) = \begin{cases} \sigma_i(n_i) & \text{if } \sigma_i \neq d_k(\iota_n), \iota_n \\ k & \text{if } \sigma_i = d_k(\iota_n) \text{ or } \iota_n. \end{cases}$$

We emphasize that f_σ need not be a morphism in Δ. However, as observed above, f_σ nevertheless induces a map $\mathrm{sd}\,\Delta^q \to \mathrm{sd}\,\Delta^n$. One can check that this map lands in $\mathrm{sd}\,\Lambda^n_k$. ■

(2) Since acyclic cofibrations are closed under (possibly infinite) compositions, it suffices to show that $X \to \mathrm{Ex}(X)$ is an acyclic cofibration. It is clear that $X \to \mathrm{Ex}(X)$ is a cofibration; indeed, Δ^n is a retract of $\mathrm{sd}\,\Delta^n$, so X is a retract of $\mathrm{Ex}(X)$. Let $i : X \to RX$ be any fibrant replacement for X (for example, $S_\bullet|X|$). If PY denotes the path space of a based simplicial set, then the "evaluation at 1" map $p : PRX \to RX$ is a fibration (we have chosen some basepoint for X and thus also for RX). Using that *sSet* is right proper, we get that the map $\tilde{i}$ in the pullback diagram

$$\begin{tikzcd} i^{-1}PRX \arrow[r, "\tilde{i}"] \arrow[d, two heads, "\tilde{p}"'] & PRX \arrow[d, two heads, "p"] \\ X \arrow[r, "i"] & RX \end{tikzcd}$$

is a weak equivalence. Thus $i^{-1}PRX \simeq *$ and we have a fibre sequence

$$\Omega RX \to i^{-1}PRX \xrightarrow{\tilde{p}} X.$$

Since Ex preserves fibrations (3) and pullbacks (4), we get a comparison of fibre sequences

$$\begin{tikzcd} \Omega RX \arrow[r] \arrow[d, "j"] & i^{-1}PRX \arrow[r, two heads, "\tilde{p}"] \arrow[d, "j"] & X \arrow[d, "j"] \\ \mathrm{Ex}\,\Omega RX \arrow[r] & \mathrm{Ex}\, i^{-1}PRX \arrow[r, two heads, "\mathrm{Ex}\,\tilde{p}"] & \mathrm{Ex}\, X. \end{tikzcd}$$

At this point, we would like to use the long exact sequence in homotopy groups to argue inductively, but since not all of the simplicial sets appearing in the diagram are fibrant, this option is not available to us. To get around this, we pass to geometric realizations and use Quillen's result that the realization of a Kan fibration is a Serre fibration to obtain an analogous diagram

$$\begin{tikzcd} |\Omega RX| \arrow[r] \arrow[d, "j"] & |i^{-1}PRX| \arrow[r, two heads, "|\tilde{p}|"] \arrow[d, "j"] & |X| \arrow[d, "j"] \\ |\mathrm{Ex}\,\Omega RX| \arrow[r] & |\mathrm{Ex}\, i^{-1}PRX| \arrow[r, two heads, "|\mathrm{Ex}\,\tilde{p}|"] & |\mathrm{Ex}\, X|. \end{tikzcd}$$

of fibration sequences in topology.

Lemma 2. *If* $Y \simeq *$ *then* $\mathrm{Ex}(Y) \simeq *$.

We will prove the lemma below, but first we use the lemma to finish the above argument. One reduces by induction to showing that $\pi_0(|X|) \cong \pi_0(|\mathrm{Ex}(X)|)$. But for any simplicial set Y, $\pi_0(|Y|) \cong \pi_0(Y) = Y_0/(\sim)$, where $\sim$ is the equivalence relation on Y_0 generated by $\overset{h}{\sim}$, where $x \overset{h}{\sim} y$ if there exists $\alpha \in Y_1$ with $d_0(\alpha) = y$ and $d_1(\alpha) = x$. But, as we have seen, a 1-simplex of $\mathrm{Ex}\, X$ is simply a zig-zag

$$x_0 \to x_1 \leftarrow x_2$$

of 1-simplices in X, so the relation $\sim$ on $\mathrm{Ex}(X)_0 = X_0$ and the relation on X_0 coincide.

Proof of Lemma 2. We have already seen that $\pi_0(Y) \cong \pi_0(\mathrm{Ex}(Y))$. We claim that $\pi_1(\mathrm{Ex}(Y)) = 0$ as well. The fundamental group $\pi_1(\mathrm{Ex}(Y)) = \pi_1(|\mathrm{Ex}(Y)|)$ has generators strings of 1-simplices of $\mathrm{Ex}(Y)$ satisfying appropriate endpoint conditions. Let $0 \xrightarrow{\alpha} 1 \xleftarrow{\beta} 2$ be an element of $\mathrm{Ex}(Y)_1$. But it is not difficult to see that

$$[0 \xrightarrow{\alpha} 1 \xleftarrow{\beta} 2] = [0 \xrightarrow{\alpha} 1 \xleftarrow{1} 1] \cdot [1 \xrightarrow{1} 1 \xleftarrow{\beta} 2] = [0 \xrightarrow{\alpha} 1 \xleftarrow{1} 1] \cdot [2 \xrightarrow{\beta} 1 \xleftarrow{1} 1]^{-1}.$$

This last element is the image of $[\alpha] \cdot [\beta]^{-1}$ from Y. Thus we have in fact shown that $\pi_1(Y) \twoheadrightarrow \pi_1(\mathrm{Ex}(Y))$. But $\pi_1(Y) = 0$, so $\pi_1(\mathrm{Ex}(Y)) = 0$. It now suffices, by the Hurewicz theorem, to show that $\mathrm{H}_*(\mathrm{Ex}(Y); \mathbb{Z}) = 0$. Using an acyclic models argument, one can show the stronger statement that

$$j_* : \mathrm{H}_*(Y; \mathbb{Z}) \cong \mathrm{H}_*(\mathrm{Ex}(Y); \mathbb{Z}).$$

In applying the acyclic models argument, one uses that any simplex $\sigma : \Delta^n \to \mathrm{Ex}(Y)$ factors through $\mathrm{Ex}(\tilde{\sigma}) : \mathrm{Ex}(\mathrm{sd}(\Delta^n)) \to \mathrm{Ex}(Y)$. One also needs that $\mathrm{Ex}(\mathrm{sd}(\Delta^n)) \simeq *$, but this follows since $\mathrm{sd}(\Delta^n)$ is homotopy equivalent to Δ^n and Ex preserves products (4). ■

(3) We first show that Ex preserves fibrations. By adjointness, it suffices to show that the subdivision functor takes the generating acyclic cofibrations $\Lambda^n_k \hookrightarrow \Delta^n$ to acyclic cofibrations. It is clear that $\mathrm{sd}\,\Lambda^n_k \hookrightarrow \mathrm{sd}\,\Delta^n$ is a cofibration(=monomorphism). Moreover, the geometric realizations are contractible, so the map in question is necessarily a weak equivalence. That Ex^∞ preserves fibrations now follows from smallness of Λ^n_k and Δ^n.

(4) Ex preserves all small limits because it has a left adjoint. The conlusion follows since finite limits commute with filtered colimits.

(5) It is clear that Ex preserves 0-simplices since $\mathrm{sd}\,\Delta^0 \cong \Delta^0$. It follows that Ex^∞ also preserves 0-simplices.

5. Comparison with $S_\bullet|-|$

The functor $S_\bullet|-| : sSet \to sSet$ is also a fibrant replacement functor, and it is certainly easier to describe than Ex^∞. In addition, $S_\bullet|-|$ preserves fibrations and finite limits, like Ex^∞. However, $S_\bullet|-|$ certainly does not preserve 0-simplices. In general $S_\bullet|X|$ will be much bigger than $\mathrm{Ex}^\infty(X)$. In addition, Ex^∞ has the advantage that one does not need to use the category of spaces in order to define it; this makes it more transportable to other contexts.