We consider functors :

$$\Gamma \xrightarrow{\ F\ } \underbrace{Mod_R \text{ or } ch_{\geq 0} Mod_R}_{\mathcal{A}}$$

"Γ- modules"

and a certain class of n-cubes in Γ : $\left(\mathcal{P}(n) \xrightarrow{\ \mathcal{X}\ } \Gamma \right)$

<u>ex</u> $X_1, X_2, X_3 \in \Gamma$

$$\begin{tikzcd}
X_1 \vee X_2 \vee X_2 \arrow[rr] \arrow[dr] \arrow[dd] & & X_1 \vee X_2 \arrow[dr, dash] \arrow[dd, dash] & \\
& X_2 \vee X_3 \arrow[rr, dash] \arrow[dd] & & X_2 \arrow[dd] \\
X_1 \vee X_3 \arrow[rr, dash] \arrow[dr] & & X_1 \arrow[dr] & \\
& X_3 \arrow[rr] & & *
\end{tikzcd}$$

Def "F is of degree (≤) n" means that F takes such cubes to (homotopy) Cartesian cubes.

Q: If F "preserves" such n-cubes, $\mathcal{X}$ does F necessarily preserve all strongly coCartesian n-cubes?

A: I Don't know, but if so, then preserving such n-cubes w/ $X_1 = X_2 = \cdots = X_n$ is enough.

Def $cr_{n+1}F(X_1, \ldots, X_n)$ = total (homotopy) fiber of $F \circ \mathcal{X}(X_1, \ldots, X_n)$.

Note total fiber of a cube measures its failure to be Cartesian so

$$cr_{n+1}F = 0 \iff \deg F \leq n$$

ex

$$\begin{tikzcd}
cr_{n+1}F(X_1,X_2) \arrow[dr] & & \\
& F(X_1 \vee X_2) \arrow[r] \arrow[d] \arrow[dr] & F(X_1) \arrow[d] \\
& F(X_2) \arrow[r] & F(*)
\end{tikzcd}$$

$\lim$

Def $cr_n F(X_1, \ldots, X_n) =: cr_n F(X) =: \perp_n F(X)$

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recursive characterization

$$cr_\phi F(X) = cr_0 F(X) \equiv F(*)$$

~~Lemma~~ $$\bigoplus_{U \subseteq n} cr_U F(X) = F(\vee_n X)$$

<u>ex</u> $F : \Gamma \longrightarrow \mathcal{M}od_R$

$$X \longmapsto \tilde{R}[X \wedge X]$$

$$\left(\quad R[*] \hookrightarrow R[X] \longrightarrow \tilde{R}[X] \quad \right)$$

Compute cross effects of F

$\underline{n=0}$ $\quad Cr_0 F \equiv F(*) = \tilde{R}[* \wedge *] = 0$

$\underline{n=1} = \{0\}$

$$cr_\phi F = 0$$

$$cr_{\{0\}} F(x) \oplus \cancel{cr_\phi F}^{\,0} = F(x) = \tilde{R}[X \wedge X]$$

$\underline{n=2 = \{0,1\}}$

$$cr_\phi F = 0$$

$$cr_{\{0\}} F(x) = cr_{\{1\}} F(x) = F(x) = \tilde{R}[X \wedge X]$$

$$\underset{\underline{\{0,1\}}\atop 2}{cr} F(x) \oplus cr_{\{0\}} F(x) \oplus cr_{\{1\}} F(x) + \cancel{cr_\phi F(x)}^{\,0} = F(X_1 \vee X_2)$$

$$F(X_1) \oplus F(X_2) \oplus$$

$$\tilde{R}[X_1 \wedge X_2] \oplus \tilde{R}[X_2 \wedge X_1]$$

$$\therefore cr_2 F(X) = 2\tilde{R}[X_1 \wedge X_2]$$

$\underline{\text{Note}}$ $\quad f(x) = x^2 \qquad f(x+y) = x^2 + y^2 + 2xy$

$$= f(x) + f(y) + cr_2 f(x)$$

$n = 3 = \{0,1,2\}$

$$F(X \vee Y \vee Z) = \tilde{R}[(X \vee Y \vee Z) \wedge (X \vee Y \vee Z)]$$

$$= \tilde{R}[X \wedge X] \oplus \tilde{R}[Y \wedge Y] \oplus \tilde{R}[Z \wedge Z]$$

$$\oplus \tilde{R}[X \wedge Y] \oplus \tilde{R}[Y \wedge X] \oplus XZ \quad ZX \quad \cancel{YZ} \quad \cancel{ZY}$$

$$= cr_0 F \oplus cr_1 F(x) \oplus cr_1 F(Y) \oplus cr_1 F(Z) \oplus$$

$$cr_2 F(x) \oplus cr_2 F(x) \oplus cr_2 F(X)$$

$$\therefore cr_3 F(x) = 0$$

$$\therefore \deg F = 2 .$$

Category Theory

$$C \underset{R}{\overset{\cancel{F}L}{\rightleftarrows}} \mathcal{D} \qquad \text{adj}$$

"$\perp$"

$$LR \xrightarrow{\varepsilon} I_{\mathcal{D}}$$

Then, we get an aug simp obj in $\mathcal{D}$ for each $D \in \mathcal{D}$:

$$D \xleftarrow{\varepsilon} \perp D \underset{\perp\varepsilon}{\overset{\varepsilon\perp}{\Longleftarrow}} \perp\perp D \overset{\varepsilon\perp\perp}{\Lleftarrow} \perp\perp\perp D \cdots$$

If $\mathcal{D}$ is abelian, then we can apply Dold-Kan to get an aug chain complex in $\mathcal{D}$:

$$D \xleftarrow{\varepsilon} \perp D \xleftarrow{\partial_1} \perp\perp D \longleftarrow \cdots$$

Back to cross effects

Note:

$$\begin{tikzcd} cr_n F(X) \arrow[r, hook] \arrow[dr, "\varepsilon"'] & F(\bigvee_n X) \arrow[d, "F(+)"] \\ & F(X) \end{tikzcd}$$

Fact: We can apply the categorical machinery to get a aug chain complex of Γ-~~fact~~modules:

$$\underbrace{F \xleftarrow{\varepsilon_F} cr_n F \leftarrow cr_n cr_n F \leftarrow \cdots}$$

$$\| \text{ def}$$

$$P_{n-1} F$$

Fact P_n has all the properties for a Taylor Tower.

$$F \longrightarrow P_n F$$

~~ex Id~~ ~~is~~ ~~Id → P_n~~ ~~is universal~~

~~among~~ is $\deg P_n F = n$ and

There is a natural map

$$F \longrightarrow P_n F$$

which is universal among functors

from F to a deg n functor.

ex $\mathcal{C} = \mathrm{Mod}_R$, $\perp M = R[M]$ ← set

$$\varepsilon_M : \perp M \longrightarrow M$$
$$R[M] \longrightarrow M$$
$$m \longmapsto m$$

$$\delta_M : \perp M \longrightarrow \perp\perp M$$
$$R[M] \hookrightarrow R[\underbrace{R[M]}_{\text{set}}]$$

Given a cotriple in $\mathcal{C}$, we can construct, for each object $c \in \mathcal{C}$, an augmented simplicial object in $\mathcal{C}$:

$$\begin{tikzcd} C & \perp C \arrow[l, "\varepsilon"'] \arrow[r, "\delta"] & \perp\perp C \arrow[l, shift right=2, "\perp\varepsilon"'] \arrow[l, shift left=2, "\varepsilon\perp"] & \cdots \arrow[l, shift right=2] \arrow[l] \arrow[l, shift left=2] \end{tikzcd} \qquad (*)$$

<u>Prop</u> If $C = \perp C'$ for some C' and if $\mathcal{C}$ is Abelian then $(*)$ is aspherical.

Cotriples

Def $\mathcal{C}$ category. A cotriple on $\mathcal{C}$ consists of:

$\perp : \mathcal{C} \longrightarrow \mathcal{C}$ endo functor

$\varepsilon : \perp \Longrightarrow \mathrm{Id}$ nat trans

$\delta : \perp \Longrightarrow \perp\perp$ nat trans

such that

$$\begin{tikzcd}
 & \perp \arrow[dl, equal] \arrow[d, "\delta"] \arrow[dr] & \\
\perp & \perp\perp \arrow[l, "\varepsilon_\perp"] \arrow[r, "\perp\varepsilon"'] & \perp
\end{tikzcd}$$

and

$$\begin{tikzcd}
\perp \arrow[r, "\delta"] \arrow[d, "\delta"'] & \perp\perp \arrow[d, "\perp\delta"] \\
\perp\perp \arrow[r, "\delta_\perp"'] & \perp\perp\perp
\end{tikzcd}$$

More Interesting : Intuition

Q: why define P_nF this way?

Idea • want functor $\Gamma \xrightarrow{P_nF} ch_{\geq 0} Mod_R$

(homotopy functor) which is deg n

(ie $cr_{n+1}((P_nF)_K) = 0 \quad \forall K$)

<u>1st attempt</u>

kill this off

$$\begin{tikzcd}
cr_nF(X) \arrow[r] \arrow[d] \arrow[dr, "\varepsilon"] & F(\vee X) \arrow[d] & \\
? \arrow[r] & F(X) \arrow[r] & H_0 \\
 & & \text{deg } n
\end{tikzcd}$$

<u>note</u>: the chain complex

$$0 \longrightarrow cr_nF(X) \xrightarrow{\varepsilon} F(X) \quad \text{has}$$

0th homology of deg n.

<u>Problem</u>: 1^{st} homology is $\ker \varepsilon$, not ness of degree n.

$$
\begin{tikzcd}
0 \arrow[r] & cr_n(cr_n F) \arrow[r, "\partial"] \arrow[d, "??"'] & cr_n F(x) \arrow[r, "\varepsilon_F"] & F(x) \\
& cr_n(\ker \varepsilon) \arrow[r, "\varepsilon_{\ker \varepsilon}"'] & \ker \varepsilon \arrow[u, hook] \arrow[r] & \begin{matrix} H_0 \\ \deg n \end{matrix}
\end{tikzcd}
$$

Then $H_1(\qquad) = \deg n$