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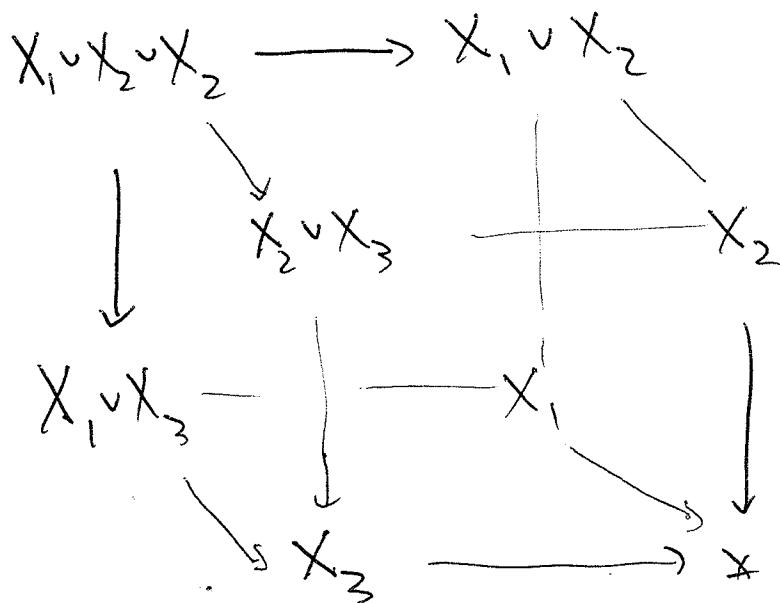
We consider functions :

$$\Gamma \xrightarrow{F} \underbrace{\text{Mod}_R \text{ or } \text{ch}_{\geq 0} \text{Mod}_R}_*$$

" Γ -modules"

and a certain class of n -cubes
in Γ : $(P(n) \xrightarrow{\chi} \Gamma)$

ex $X_1, X_2, X_3 \in \Gamma$



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Def "F is of degree ($\leq$) n" means that
F takes such cubes to (homotopy)
Cartesian cubes.

"X"

Q: If F "preserves" such n-cubes,
does F necessarily preserve all
strongly coCartesian n-cubes?

A: I Don't know, but if ~~it~~ so, then
preserving such n-cubes w/ $X_1 = X_2 = \dots = X_n$
is enough.

Def $cr_{w+1} F(X_1, \dots, X_n) = \text{total (homotopy)}$
fiber of $F \circ X(X_1, \dots, X_n)$.

Note total fiber of a cube measures
its failure to be Cartesian so

$$cr_n F = 0 \iff \deg F \leq n$$

ex

$$\begin{array}{ccccc} & cr_n F(x_1, x_2) & & & \\ & \downarrow & & & \\ & F(x_1, x_2) & \longrightarrow & F(x_1) & \\ & \downarrow & \searrow & \nearrow & \downarrow \\ & & \text{diag} & & \\ & F(x_2) & \longrightarrow & F(*) & \end{array}$$

Def $cr_n F(x_1, \dots, x_n) =: cr_n F(x) =: \perp_n F(x)$

recursive characterization

$$cr_\emptyset F(x) = cr_0 F(x) = F(*)$$

LEM $\bigoplus_{u \leq n} cr_u F(x) = F(\bigvee_n x)$

(4)

$$\underline{\text{ex}} \quad F : \Gamma \longrightarrow \text{Mod}_R$$

$$X \longmapsto \tilde{R}[X \wedge X]$$

$$(R[*] \longleftrightarrow R[X] \longrightarrow \tilde{R}[X])$$

Compute cross effects of F

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$$\underline{n=0} \quad C_0 F \equiv F(*) = \tilde{R}[* \wedge *] = 0$$

$$\underline{n=1} = \{0\}$$

$$C_\emptyset F = 0$$

$$C_{\{0\}} F(x) \oplus \cancel{C_\emptyset F} = F(x) = \tilde{R}[x \wedge x]$$

$$\underline{n=2} = \{0, 1\}$$

$$C_\emptyset F = 0$$

$$C_{\{0\}} F(x) = C_{\{1\}} F(x) = F(x) = \tilde{R}[x \wedge x]$$

$$C_{\frac{\{0,1\}}{2}} F(x) \oplus C_{\{0\}} F(x) \oplus C_{\{1\}} F(x) + \cancel{C_\emptyset F(x)} = F(x_1 \wedge x_2)$$

$$\begin{aligned} & (F(x_1) \oplus F(x_2)) \oplus \\ & \quad \tilde{R}[x_1 \wedge x_2] \oplus \tilde{R}[x_2 \wedge x_1] \end{aligned}$$

$$\therefore C_2 F(x) = 2\tilde{R}[x_1 \wedge x_2]$$

Note

$$f(x) = x^2$$

$$f(x+y) = x^2 + y^2 + 2xy$$

$$= f(x) + f(y) + 2C_2 f(x)$$

$$n=3=\{0,1,2\}$$

5

$$F(x \vee y \vee z) = \tilde{R}[(x \vee y \vee z) \wedge (x \wedge y \vee z)]$$

$$= \tilde{R}[x \wedge x] \oplus \tilde{R}[y \wedge y] \oplus \tilde{R}[z \wedge z]$$

$$\oplus \tilde{R}[x \wedge y] \oplus \tilde{R}[y \wedge x] \oplus xz \vee zx \vee zy \vee yz$$

$$= \alpha_0 F \oplus \alpha_1 F(x) \oplus \alpha_1 F(y) \oplus \alpha_1 F(z) \oplus$$

$$\alpha_2 F(x) \oplus \alpha_2 F(y) \oplus \alpha_2 F(x)$$

$$\therefore \alpha_3 F(x) = 0$$

$$\therefore \deg F = 2$$

Category Theory

(6)

$$\begin{array}{ccc} \mathcal{C} & \begin{array}{c} \xrightarrow{F_L} \\ \xleftarrow{R} \end{array} & \mathcal{D} \end{array} \quad \text{adj}$$

"1"

$$\textcircled{LR} \xrightarrow{\epsilon} I_{\mathcal{D}}$$

Then, we get an ^{for each $D \in \mathcal{D}$} any simp obj in $\mathcal{D}$:

$$D \xleftarrow{\epsilon} ID \begin{array}{c} \xleftarrow{\epsilon I} \\ \xrightarrow{I \epsilon} \end{array} IID \begin{array}{c} \xleftarrow{\epsilon II} \\ \xrightarrow{I II} \end{array} IIIID \dots$$

If $\mathcal{D}$ is abelian, then we can apply
Dold-Kan to get an any chain
complex in $\mathcal{D}$:

$$D \xleftarrow{\epsilon} ID \xleftarrow{d_1} IID \leftarrow \dots$$

Back to cross effects

(7)

Note:

$$\begin{array}{ccc}
 cr_n F(x) & \xrightarrow{\quad} & F(V_n X) \\
 & \searrow \varepsilon & \downarrow F(+ \\
 & & F(X)
 \end{array}$$

Fact: We can apply the categorical machinery to get a chain complex of Γ -~~sub~~^{aug}modules:

$$F \xleftarrow{\varepsilon_F} cr_n F \xleftarrow{\quad} cr_n cr_n F \xleftarrow{\quad} \dots$$

|| def

$$P_{n-1} F$$

Fact P_n has all the properties for a Taylor Tower.

~~among it $\deg P_n F = n$ and~~ $F \rightarrow P_n F$ is universal (8)

~~There is a natural map~~

$$F \longrightarrow P_n F$$

which is universal among factors
~~from~~ F to a $\deg n$ factor.

$$\underline{\text{ex}} \quad \mathcal{C} = \text{Mod}_R, \quad \perp M = R[M] \quad (2)$$

$\uparrow_{\text{set}}$

$$\begin{aligned} \epsilon_M : \perp M &\longrightarrow M \\ R[M] &\longrightarrow M \\ m &\longmapsto m \end{aligned}$$

$$\begin{aligned} \delta_M : \perp \perp M &\longrightarrow \perp \perp M \\ R[M] &\hookrightarrow R[\underbrace{R[M]}_{\text{set}}] \end{aligned}$$

~~*****~~

Given a cotriple in $\mathcal{C}$, we can construct, for each object $C \in \mathcal{C}$, an augmented simplicial object in $\mathcal{C}$:

$$C \xleftarrow{\epsilon} \perp C \xrightleftharpoons[\epsilon \downarrow]{\perp \epsilon} \perp \perp C \rightrightarrows \dots \quad (*)$$

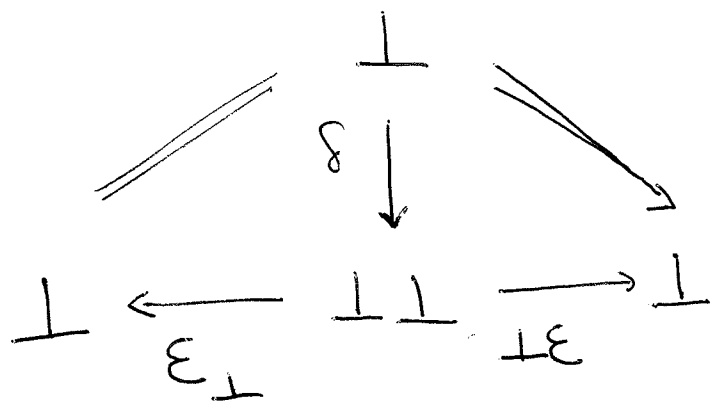
Prop If $C = \perp C'$ for some C' and if $\mathcal{C}$ is Abelian then $(*)$ is aspherical.

Cotriples

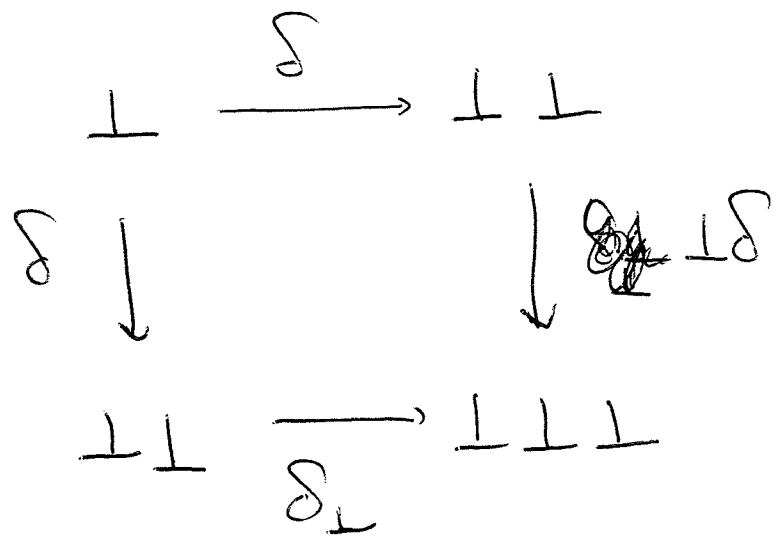
Def $\mathcal{C}$ category. A cotriple on $\mathcal{C}$ consists of :

$$\begin{aligned} \perp &: \mathcal{C} \longrightarrow \mathcal{C} && \text{endo functor} \\ \varepsilon &: \perp \Longrightarrow \text{Id} && \text{nat trans} \\ \delta &: \perp \Longrightarrow \perp \perp && \text{nat trans} \end{aligned}$$

such that



and



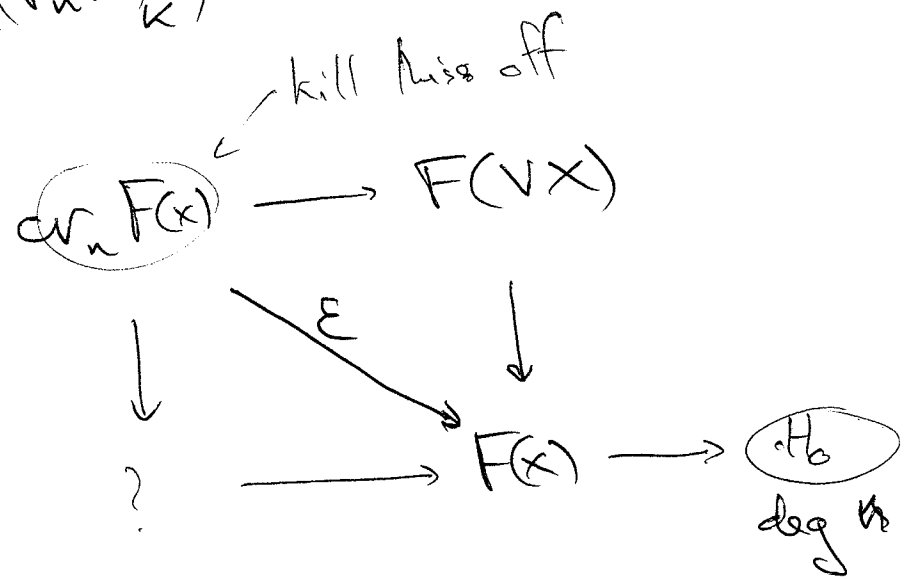
More Interesting: Induction (9)

Q: Why define $P_n F$ this way?

Idea: want functor $\Gamma \xrightarrow{P_n F} \text{ch}_{\geq 0} \text{Mod}_R$
(homopy functor) which is deg n

(ie $cr_{n+1}((P_n F)_K) = 0 \quad \forall K$)
H

1st attempt



note: the chain complex

$$0 \rightarrow cr_n F(x) \xrightarrow{\epsilon} F(x) \quad \text{has}$$

n th homology of deg ~~at~~ n .

Problem: 1st homology is $\text{Ker } E$ ~~is~~,
not necessarily of degree n .

(10)

$$\begin{array}{ccccc}
 0 \rightarrow C_n(C_n F) & \xrightarrow{\partial} & C_n F(x) & \xrightarrow{E_F} & F(x) \\
 \downarrow \text{??} & & \uparrow & & \downarrow \text{crossed out} \\
 C_n(\text{Ker } E) & \xrightarrow{E_{\text{Ker } E}} & \text{Ker } E & \xrightarrow{\text{H}_0 \text{ deg } n} & \text{crossed out}
 \end{array}$$

Then $H_1(\quad) = \text{deg } n$

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