Math 535
Homework XI

Due Mon. May 4

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Problem 1. Let $f : X \to Y$. Show that f is a homotopy equivalence if there exists maps $g, h : Y \to X$ such that $f \circ g \simeq 1_Y$ and $h \circ f \simeq 1_X$.

Problem 2. Show that if G and H are groups, then the "direct product" $G \times H$ discussed in class satisfies the universal property for products of groups.

Problem 3. Show that if (X, x) and (Y, y) are based spaces, then

$$\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y).$$

Problem 4. Let $i : A \hookrightarrow X$ be the inclusion of a subspace. A **deformation retraction** of X onto A is (1) a retract $r : X \to A$ (so $r \circ i = 1_A$) and (2) a homotopy $H : i \circ r \simeq 1_X$ such that $H(a, t) = a$ for all t.

(i) Show that there is a deformation retraction of $\mathbb{R}^n \setminus \{0\}$ onto the subspace S^{n-1}.

(ii) Show that for any $x \in S^1$, there is a retract $r : S^1 \to \{x\}$, but no deformation retraction.

Problem 5. Show that for a space X, the following conditions are equivalent:

(i) Every map $S^1 \to X$ is homotopic to a constant map.

(ii) Every map $S^1 \to X$ can be extended to a map $D^2 \to X$ (D^2 is $B_{\leq 1}(\mathbf{0})$, the closed unit disc in $\mathbb{R}^2$).

(iii) $\pi_1(X, x) \cong 0$ for all $x \in X$.

Problem 6. We can think of $S^1 \subseteq \mathbb{R}^2$ as the set of complex numbers of norm 1.

(i) Show that multiplication of complex numbers makes S^1 into a group.

(ii) This then gives another way of multiplying loops in S^1. Given loops γ_1 and γ_2 in S^1 starting at $(1, 0)$, define a loop $\gamma_1 \Box \gamma_2$ by $\gamma_1 \Box \gamma_2(t) = \gamma_1(t) \cdot \gamma_2(t)$. Show that $\gamma_1 \Box \gamma_2 \simeq_p \gamma_1 \star \gamma_2$.

Problem 7. Show that there are closed subsets W_1, W_2, W_3, and W_4 of S^2 such that their union is all of S^2 and such that no W_i contains a pair of antipodal points.