Math 535
Homework II

Due Wed. Feb. 4

Bertrand Guillou

Problem 1. (i) Let (X, d) be a metric space and let $x \in X$. Define $d_x : X \to \mathbb{R}$ by $d_x(y) = d(x, y)$. Show that d_x is continuous (X is equipped with the metric topology).

(ii) Use the above to show that

$$S^{n-1} = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2 + \cdots + x_n^2 = 1\right\} \subseteq \mathbb{R}^n$$

is closed.

Problem 2. Let $X = \mathbb{R} \cup \{z\}$. Define a proper subset to be closed if it is finite and does not contain the point z. Show that this defines a topology on X and that the point z is dense in X (such a point is called a *generic* point).

Problem 3. Let A and B denote subsets of a topological space X.

(i) Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(ii) Show that $\bigcup \overline{A_\alpha} \subset \overline{\bigcup A_\alpha}$ and give an example these two subsets are not equal.

Problem 4. Let f and g be continuous functions $X \to \mathbb{R}$. Show that the subset $W \subseteq X$ of points x such that $f(x) = g(x)$ is closed. This shows that a real-valued continuous function is determined by its values on a dense subset.

Problem 5. Suppose $f : X \to Y$ is continuous and let $A \subseteq X$ be a subset. If x is a limit point of A, is $f(x)$ a limit point of $f(A)$?

Problem 6. Show that if $W \subseteq X$ is any subset, then

$$W^o = X \setminus \overline{(X \setminus W)}.$$

Problem 7. Show that $f : X \to Y$ is continuous if and only if for every closed $Z \subseteq Y$, $f^{-1}(Z) \subseteq X$ is closed.

Problem 8. Let C and D be sets, and suppose given functions $f : C \to D$ and $g : D \to C$.

(i) Show that if $g \circ f$ is the identity function of C, then f is injective.

(ii) Show that if $g \circ f$ is the identity function of C, then g is surjective.

(iii) Use the above to conlude that f is bijective if and only if it is a (categorical) isomorphism (i.e., there exists a function $g : D \to C$ such that $f \circ g$ is the identity on D and $g \circ f$ is the identity on C).

Note that the above shows, by forgetting about the topologies, that a homeomorphism is a bijection.

Problem 9. Show that $\mathbb{R}$ is homeomorphic to $(0, 1)$. (Hint: you should be able to use functions of the form $\frac{ax+b}{cx+d}$ for appropriate choices of a, b, c, and d.)

Problem 10. (a) Let $X = \mathbb{R} \cup \{\infty\}$, topologized as follows: if a subset W does not contain the point ∞, then it is open if it is open in the usual topology on $\mathbb{R}$; if W does contain ∞, then W is open if $X \setminus W$ is closed in $\mathbb{R}$ (under the usual topology) and contained in some closed interval $[a, b]$. Show this defines a topology on X.

(b) Show that the space X from part (a) is homeomorphic to

$$S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}.$$

Problem 11. Define a function $f : [0, 1) \cup \{2\} \to [0, 1]$ to be the inclusion $[0, 1) \hookrightarrow [0, 1]$ and $f(2) = 1$. Show that f is a continuous bijection but not a homeomorphism.