Math 535
Homework III

Due Wed. Feb. 11

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Problem 1. Let (X, d) be a metric space.

(i) Show that the metric $d : X \times X \to \mathbb{R}$ is continuous.

(ii) Show that the diagonal subset $\Delta(X) \subseteq X \times X$ is closed.

(iii) Show that if Y is any topological space and f and g are continuous functions $Y \to X$, then the set of $y \in Y$ such that $f(y) = g(y)$ is closed in Y.

Problem 2. (i) Give an example of a continuous $f : \mathbb{R} \to \mathbb{R}$ which is closed but not open.

(ii) Give an example of a continuous $g : \mathbb{R} \to \mathbb{R}$ which is open but not closed.

Problem 3. Show that if A and B are sets, then the cartesian product $A \times B$, together with the usual projection functions $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ satisfies the universal property for a product; that is, show that given any other set C and functions $f : C \to A$ and $g : C \to B$, then there is a *unique* function $h : C \to A \times B$ such that $f = \pi_A \circ h$ and $g = \pi_B \circ h$.

Problem 4. Let $A_\alpha \subseteq X_\alpha$ be a subspace for each $\alpha \in \mathcal{A}$. Show that the subspace topology on $\prod_{\alpha \in \mathscr{A}} A_\alpha \subseteq \prod_{\alpha \in \mathscr{A}} X_\alpha$ coincides with the product topology (and therefore not, in general, with the box topology).

Problem 5. Let $\mathcal{Z} \subseteq \mathbb{R}^\infty$ be the subset consisting of sequences which are eventually zero (in other words, only finitely many of the entries are nonzero). Find the closure of $\mathcal{Z}$ in $\mathbb{R}^\infty$ under the box and product topologies.

Problem 6. Define $\tilde{d} : \mathbb{R}^2 \to \mathbb{R}$ by $\tilde{d}(r, s) = \min(|r - s|, 1)$. The **uniform metric** on $\mathbb{R}^\infty$ is defined by $\overline{\rho}(\mathbf{x}, \mathbf{y}) = \sup_n \tilde{d}(x_n, y_n)$. The topology induced on $\mathbb{R}^\infty$ by the uniform metric is referred to as the uniform topology.

(i) Show that the box topology on $\mathbb{R}^\infty$ is finer than the uniform topology, but the uniform is finer than the product topology.

(ii) Let $0 < \varepsilon < 1$ and $\mathbf{0} = (0, 0, 0, \dots)$ be the constant sequence. Show that the ε-ball $B_\varepsilon^{\overline{\rho}}(\mathbf{0})$ about $\mathbf{0}$ is open in the uniform but not the product topology.

(iii) Let $0 < \varepsilon < 1$. Show that $(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times \cdots \subseteq \mathbb{R}^\infty$ is open in the box topology but not the uniform topology.

Problem 7. Let $I = [0, 1]$ and $\partial I = \{0, 1\} \subseteq I$. Show that $I/\partial I \cong S^1$.