Math 535
Homework IV
(corrected)

Due Wed. Feb. 18

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Problem 1. If d is a metric on X, define $\tilde{d}$, a "bounded" version of d, by

$$\tilde{d}(x, y) = \min\{d(x, y), 1\}.$$

Let (X_1, d_1), (X_2, d_2), $\ldots$ be metric spaces. Show that

$$D(\mathbf{x}, \mathbf{y}) = \sup_n \frac{\tilde{d}(x_n, y_n)}{n}$$

defines a metric on $\prod_n X_n$. (The proof given in class for $X_n = \mathbb{R}$ generalizes to show that this metric induces the product topology, so that a (countable) product of metrizable spaces is metrizable.)

Problem 2. Let X be a set. Show that a set of functions $f_n : X \to \mathbb{R}$ converge uniformly to f if and only if f_n converges to f in the uniform topology on $\mathbb{R}^X$.

Problem 3. Let X be a space and Y a metric space. Let $f_n : X \to Y$ be continuous for each n. Suppose $\{x_n\}$ converges to $x \in X$ and also that f_n converges uniformly to f. Show that the sequence $f_n(x_n)$ converges to $f(x)$.

Problem 4. Let X be $\mathbb{R}$ but with open sets only those of the form (a, ∞), allowing $a = \pm\infty$ (convince yourself that this is a topology).

(i) For any subset $A \subseteq X$, describe the closure $\overline{A}$ of A in X. In particular, find the closure of the singleton sets $\{x\}$, and use this to show that X is not metrizable.

(ii) Show, however, that (a very strong version of) the sequence lemma holds in X by showing that given any set A, there exists a sequence in A which simultaneously converges to every point of $\overline{A}$.

(iii) Use the above to show that the only continuous functions $f : X \to \mathbb{R}$ are the constant functions.

Problem 5. A space is **totally disconnected** if the only nonempty connected subsets are the singleton sets $\{x\}$. Show that $\mathbb{Q}$ is totally disconnected.

Problem 6. Show that $\mathbb{R}^\infty$, under the box topology, is not connected by showing that the set $\mathcal{B}$ of bounded sequences is closed and open.

Problem 7. We know, in contrast, that $\mathbb{R}^\infty$, being the product of connected spaces, is connected under the product topology. So $\mathcal{B}$ cannot be both closed and open under the product topology.

(i) Show that $\mathcal{B}$ is dense in $\mathbb{R}^\infty$. (Hint: for any $\mathbf{x} \in \mathbb{R}^\infty$, find a sequence in $\mathcal{B}$ which converges to $\mathbf{x}$.)

(ii) Show that $\mathcal{B}^o$, the interior of $\mathcal{B}$, is empty.