Math 535
Homework V

Due Fri. Feb. 27

Bertrand Guillou

Problem 1. (The Cantor set) Let $C_0 = I$. For $k \geq 1$, define $C_k \subseteq I$ inductively by

$$C_k = C_{k-1} \cap \left(\left[0, \frac{1}{3^k}\right] \cup \left[\frac{2}{3^k}, \frac{3}{3^k}\right] \cup \left[\frac{4}{3^k}, \frac{5}{3^k}\right] \cup \cdots \cup \left[\frac{3^k - 1}{3^k}, 1\right] \right)$$

(so each C_k is a union of intervals, and each C_k is obtained from C_{k-1} by removing the "middle thirds" of the intervals). Define the cantor set by $C = \bigcap_k C_k$.

Show that C is totally disconnected. (Hint: what do the components of each C_k look like?)

Problem 2. Give an example of a locally connected X and a continuous $f : X \to Y$ such that $f(X)$ is not locally connected.

Problem 3. Let $X = \mathbb{N}$, equipped with the cofinite topology (see HW1, problem 7). Show that X is connected and locally connected, but not path connected or locally path connected.

Problem 4. Show that if X is Hausdorff, then limits of sequences in X are unique. That is, if $\{x_n\}$ converges to both x and y, then $x = y$.

Problem 5. (i) Show that a space X is Hausdorff if and only if the diagonal $\Delta(X) \subseteq X \times X$ is closed. By HW3, problem 1(iii), this shows that if Y is any space and f, g are continuous functions $Y \to X$ which agree on a dense subset of Y, then $f = g$.

(ii) Show that if Y is Hausdorff and $f : X \to Y$ is continuous, then the graph of f is closed in $X \times Y$.

Problem 6. A subset $A \subseteq X$ is said to be **locally closed** if it can be written as the intersection of an open set and a closed set.

(i) Show that for any space X, the diagonal $\Delta(X) \subseteq X \times X$ is always locally closed.

(Hint: Say a point $(x, y) \in X \times X$ is "bad" if x and y do not satisfy the Hausdorff property; that is, (x, y) is bad if every pair of neighborhoods U_x and U_y intersect nontrivially. Show that if $\mathcal{B}$ denotes the set of bad points, then $\overline{\Delta(X)} = \Delta(X) \cup \mathcal{B}$. Then show that $\mathcal{B}$ is closed by showing that it contains its accumulation points.)

(ii) Conclude that for a continuous map $f : X \to Y$ between arbitrary spaces X and Y, the graph of f is locally closed in $X \times Y$.

Problem 7. (The line with doubled origin) Let Y be the quotient of $\mathbb{R} \times \{-1, 1\}$ by the relation $(x, -1) \sim (x, 1)$ if $x \neq 0$.

(i) Show that Y is not Hausdorff.

(ii) Define $f, g : \mathbb{R} \to Y$ by $f(x) = (x, 1)$ and $g(x) = (x, -1)$. Show that f and g are both continuous. Note, however, that they agree on the dense subset $\mathbb{R} \setminus \{0\}$ but are clearly not equal.