Math 535
Homework VI

Due Fri. Mar. 6

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Problem 1. (i) Show that the Cantor set C (HW5, problem 1) is compact.

(ii) Show that any compact, locally connected space has finitely many components. Conclude that the Cantor set is not locally connected.

(iii) Any $x \in I$ has a "ternary" expansion; that is, any x can be written $x = \sum_{i \geq 1} \frac{x_i}{3^i}$, where $x_i \in \{0, 1, 2\}$ for all i. Show that the function

$$\{0, 2\}^\infty \to I$$

defined by

$$(x_1, x_2, x_3, \dots) \mapsto \sum_{i \geq 1} \frac{x_i}{3^i}$$

induces a homeomorphism $\{0, 2\}^\infty \cong C$.

Problem 2. Show the tube lemma fails for noncompact spaces by giving an open set $N \subset (0, \infty) \times \mathbb{R}$ which contains $(0, \infty) \times \{0\}$ but such that there is no neighborhood V of 0 in $\mathbb{R}$ for which $(0, \infty) \times V \subset N$.

Problem 3. (i) Show that the only topology on a finite set which makes the space Hausdorff is the discrete topology.

(ii) Show more generally that if τ_1 and τ_2 are topologies on the same space X such that τ_1 is finer than τ_2 and such that both (X, τ_1) and (X, τ_2) are compact Hausdorff, then $\tau_1 = \tau_2$.

Problem 4. Generalize the proof given in class of the statement that compact subsets of Hausdorff spaces are closed to show that if X is Hausdorff and A and B are disjoint compact subsets of X, then there exist disjoint open sets U and V containing A and B.

Problem 5. (i) Show that if Y is compact, then the projection $\pi_X : X \times Y \to X$ is closed for any X.

(ii) Give an example in which the other projection π_Y is not closed.

Problem 6. A subset A of a metric space X is said to be **bounded** if there exists $M > 0$ such that $d(a_1, a_2) < M$ for all a_1 and a_2 in A.

(i) Show that a compact subset of a metric space is closed and bounded.

(ii) Let X be $\mathbb{R}$ equipped with the bounded version of the standard metric. Show that there are closed sets which are bounded in this metric but not compact.

(Although closedness and compactness only depend on the topology, boundedness depends on a choice of a metric inducing the topology and is thus not a well-defined topological invariant).