Math 535
Homework VII

Due Fri. Mar. 20

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Problem 1. A space X is **first-countable** if for each $x \in X$, there is a countable set $\mathscr{B}$ of neighborhoods of x such that any neighborhood U of x contains some $B \in \mathscr{B}$.

(i) Show that if X is first-countable, $A \subseteq X$ is a subset, and $x \in \overline{A}$, then there is a sequence in A converging to x.

(ii) Show that if X is first-countable and Y is arbitrary, then a function $f : X \to Y$ is continuous if it takes convergent sequences in X to convergent sequences in Y.

Problem 2. Show that a *closed* subset of a normal space is normal.

Problem 3. Show that if $\prod_\alpha X_\alpha$ is normal, then each X_α is normal.

Problem 4. (The Tangent Disc Topology) Let

$$X = \{(x, y) \in \mathbb{R}^2 | y \geq 0\}$$

be the upper half-plane, equipped with the following topology. Let $H \subseteq X$ be the points with y-coordinate > 0 and let L denote the x-axis. A subset of H is open in X if and only if it is open in the usual topology on $\mathbb{R}^2$. For each $(x, 0) \in L$, add a basis element of the form $\{(x, 0)\} \cup D$, where D is an open disc in H which is tangent to L at $(x, 0)$.

(i) Show that every subset of L is closed. Show more generally that a closed subset $A \subseteq X$ is of the form $(\overline{A \cap H}) \cup B$, where the closure is taken in $\mathbb{R}^2$ (with the usual topology) and B is an arbitrary subset of L.

(ii) Show that X is regular.

(Hint: If $(x, 0) \in L$ and A is closed not containing x, let $\{(x, 0)\} \cup D$ be a basis element containing $(x, 0)$ and disjoint from A. Then $D = B_\delta(x, \delta)$. Let $D' = B_{\delta/2}(x, \delta/2)$ and $D'' = B_{3\delta/4}(x, 3\delta/4)$. Then $A \subseteq X \setminus \overline{D''}$.)

We will see later that this space is not normal