Math 535
Homework VIII

Due Mon. Apr. 6

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Problem 1. A space Y is said to have the **universal extension property** (UEP) if for every normal X, closed $A \subseteq X$, and continuous $f : A \to Y$, there is a continuous extension $\tilde{f} : X \to Y$ of f.

(i) A subspace $W \subseteq Z$ is said to be a **retract** of Z if there is a continuous function $r : Z \to W$ such that $r(w) = w$ for all $w \in W$. Show that if Z is a retract of Y, then Z satisfies the UEP if Y does.

(ii) Let $Y_\alpha \neq \emptyset$ for all $\alpha \in \mathscr{A}$. Show that the spaces Y_α satisfy the UEP if and only if $\prod_\alpha Y_\alpha$ satisfies the UEP.

Problem 2. Let Y be Hausdorff and $Z \hookrightarrow Y$ a retract of Y. Show that Z is closed in Y.

Problem 3. Show that for any indexing set J, the space I^J is normal ($I = [0, 1]$).

Problem 4. Show that any connected normal space with at least two points is uncountable.

Problem 5. Let X be a metrizable space. Show that the following are equivalent.

(i) X is bounded under every metric that induces the given topology on X.

(ii) Every continuous function $f : X \to \mathbb{R}$ is bounded.

(iii) X is compact.

(Hint: By "everyone's favorite result about metric spaces", (iii) is equivalent to requiring X to be sequentially compact. So for (ii)⇒(iii), suppose $\{x_n\}$ is a sequence with no convergent subsequence. Use this to find a closed, discrete, countably infinite subset $A \subseteq X$. Then there is some bijection $\mathbb{N} \to A$, $n \mapsto a_n$. The inverse $a_n \mapsto n$ is then a continuous, unbounded function on A. Finally, extend this to a function on X.

For (i)⇒(ii), let $f : X \to \mathbb{R}$ be any continuous function. Define $\phi : X \to X \times \mathbb{R}$ by $\phi(x) = (x, f(x))$. Let d be a metric inducing the topology on X and define a metric D on $X \times \mathbb{R}$ by $D((x, r), (y, s)) = \max\{d(x, y), |r - s|\}$.

Show this induces the product topology on $X \times \mathbb{R}$. Show that D yields a metric on X and use this to conclude that f must be bounded.)

Problem 6. Let X be completely regular. Show that X is connected if and only if its Stone-Čech compactification $\beta(X)$ is connected.

Problem 7. Let X be a discrete space.

(i) Show that if $A \subset X$ is a proper subset, then the closures of A and $X \setminus A$ in $\beta(X)$ are disjoint.

(ii) Show that if U is open in $\beta(X)$, then $\overline{U}$ is open in $\beta(X)$