Math 432
Homework X

Due Fri. Apr. 23

Problem 1. Let $\mathscr{B}$ be a basis for a topology on X and define a subset $U \subseteq X$ to be open, as it was in class, if for every $x \in U$ there is $V \in \mathscr{B}$ such that $x \in V \subseteq U$. Show that this satisfies the definition of a topology.

Problem 2. (a) (The cofinite topology) Let X be any set and define a nonempty subset $U \subseteq X$ to be open if $X \setminus U$ is finite. Show that this defines a topology on X.

(b) Let $X = \mathbb{R} \cup \{z\}$. Define a proper subset to be closed if it is finite and does not contain the point z. Show that this defines a topology on X and that the point z is dense in X (such a point is called a generic point).

Problem 3. Suppose $f : X \to Y$ is continuous and let $A \subseteq X$ be a subset. If x is a limit point of A, is $f(x)$ a limit point of $f(A)$?

Problem 4. (a) Let $X = \mathbb{R} \cup \{\infty\}$, topologized as follows: if a subset W does not contain the point ∞, then it is open if it is open in the usual topology on $\mathbb{R}$; if W does contain ∞, then W is open if $X \setminus W$ is closed in $\mathbb{R}$ (under the usual topology) and contained in some closed interval $[a, b]$. Show this defines a topology on X.

(b) (**Deferred until next week's homework.**) ~~Show that the space X from part (a) is homeomorphic to~~

$$S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}.$$

Problem 5. Define a function $f : [0, 1) \cup \{2\} \to [0, 1]$ to be the inclusion $[0, 1) \hookrightarrow [0, 1]$ and $f(2) = 1$. Show that f is a continuous bijection but not a homeomorphism.

Problem 6. Let X be $\mathbb{R}$ but with open sets only those of the form (a, ∞), allowing $a = \pm\infty$ (convince yourself that this is a topology).

(i) For any subset $A \subseteq X$, describe the closure $\overline{A}$.

(ii) Show that X is not Hausdorff, and show that if $W \subseteq X$ is closed, there exists a sequence in W that converges to every point of W.

Problem 7. A space is **totally disconnected** if the only nonempty connected subsets are the singleton sets $\{x\}$. Show that $\mathbb{Q}$ is totally disconnected.