Math 432
Homework XI

Due Fri. Apr. 30

Problem 1. (Deferred from last week) Show that the space X from part 4(a) of Homework X is homeomorphic to

$$S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}.$$

Problem 2. Let $X = \mathbb{N}$, equipped with the cofinite topology (see HW X, problem 2(a)).

(a) Show that X is compact.

(b) Show that X is connected but not path connected. (You may use the fact that $[0, 1]$ cannot be written as a countably infinite disjoint union of closed subsets.)

Problem 3. Generalize the proof given in class of the statement that compact subsets of Hausdorff spaces are closed to show that if X is Hausdorff and A and B are disjoint compact subsets of X, then there exist disjoint open sets U and V containing A and B.

Problem 4. (i) Show that a space X is Hausdorff if and only if the diagonal $\Delta(X) \subseteq X \times X$ is closed.

(ii) Show that if Y is Hausdorff and $f : X \to Y$ is continuous, then the graph of f is closed in $X \times Y$.

Problem 5. Show that $\mathbb{R}^\infty$, under the box topology, is not connected by showing that the set $\mathcal{B}$ of bounded sequences is closed and open.

Problem 6. We know, in contrast, that $\mathbb{R}^\infty$, being the product of connected spaces, is connected under the product topology. So $\mathcal{B}$ cannot be both closed and open under the product topology.

(i) Show that $\mathcal{B}$ is dense in $\mathbb{R}^\infty$. (Hint: for any $\mathbf{x} \in \mathbb{R}^\infty$, find a sequence in $\mathcal{B}$ which converges to $\mathbf{x}$.)

(ii) Show that $\mathcal{B}^o$, the interior of $\mathcal{B}$, is empty.