CLASS NOTES
MATH 527 (SPRING 2011)
WEEK 11

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1. Mon, Apr. 4

Now that we understand cohomology as classifed by Eilenberg-Mac Lane spaces, we can discuss a process dual to that of CW approximation.

Let X be a space. A **Postnikov tower** for X is a (homotopy commutative) diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & P_2(X) \arrow[d, "q_1"] \\
X \arrow[uur] \arrow[ur, "f_2"'] \arrow[r, "f_1"'] \arrow[dr, "f_0"'] & P_1(X) \arrow[d, "q_0"] \\
 & P_0(X)
\end{tikzcd}$$

such that

(1) For each i, the map $\pi_j(f_i) : \pi_j(X) \longrightarrow \pi_j(P_i(X))$ is an isomorphism for $j \leq i$
(2) $\pi_j(P_i(X)) = 0$ for $j > i$.

Note that it follows that the map $\pi_j(P_{i+1}(X)) \xrightarrow{\pi_j(q_i)} \pi_j(P_i(X))$ is an isomorphism for $j \neq i+1$ and is the zero map when $j = i+1$. We also deduce that the fiber

$$F(q_i) \longrightarrow P_{i+1}(X) \longrightarrow P_i(X)$$

is an Eilenberg-Mac Lane space $F(q_i) \simeq K(\pi_{i+1}(X)), i+1)$.

We give two slightly different cosntructions of Postnikov towers.

In either case, for each i, we build $X \longrightarrow P_i(X)$ as a relative CW complex. We start by setting $P_i(X)_{i+1} = X$ and we attach cells in dimensions $i+2$ and higher.

Version I: (small) Choose a set of generators for $\pi_{i+1}(X)$, and for each chosen generator, we attach an $i+2$-cell to X to kill that element of homotopy. We call the result $P_i(X)_{i+2}$. Note that $P_i(X)_{i+2}$ has the correct homotopy in degrees $\leq i+1$. Now choose generators for $\pi_{i+2}(P_i(X)_{i+2})$ and attach $i+3$-cells to annihilate this homotopy group. Proceeding inductively, we have produced $X \xrightarrow{f_i} P_i(X)$ with the desired properties.

Version II: (functorial) To get something functorial, you should avoid making any choices. So we attach an $i+2$-cell to $P_i(X)_{i+1} = X$ for <u>every</u> map $S^{i+1} \longrightarrow P_i(X)_{i+1}$. At the next stage, we attach an $i+3$-cell for every map from an S^{i+2}, and so on. The resulting relative CW complex has many more cells but still satisfies the desired homotopy properties.

We have yet to specify the maps $q_i : P_{i+1}(X) \longrightarrow P_i(X)$ in the tower. We define q_i as a cellular map, rel X. Recall that $P_{i+1}(X)_{i+1} = X = P_i(X)_{i+1}$, so we may take q_i on the $(i+1)$-skeleton to be the identity map. On the $(i+2)$-skeleton, the map $P_{i+1}(X)_{i+2} = X \longrightarrow P_i(X)_{i+2}$ is the inclusion. The $(i+3)$-skeleton is the first one requiring any work. Let $\Phi : D^{i+3} \longrightarrow P_{i+1}(X)_{i+3}$ be a cell with attaching map $\varphi : S^{i+2} \longrightarrow P_{i+1}(X)_{i+2} = X$.

Version I: The element $q_i \circ \alpha$ is null in $\pi_{i+2}(P_i(X)_{i+2})$, so we may take any extension over the disk as the definition of q_i on Φ.

Version II: The composite $q_i \circ \alpha$ is the attaching map for a $(i+3)$-cell of $P_i(X)_{i+2}$, so we can use the inclusion of this cell as the definition of q_i on the cell Φ. Note that no choice was made here.

The same argument allows us to define q_i on the higher skeleta. Using either definition, we get $q_i \circ f_{i+1} = f_i$.

Functoriality. Suppose given a maps $g : X \longrightarrow Y$ and $h : Y \longrightarrow Z$.

Using Version I, we can build maps $g_i : P_i(X) \longrightarrow P_i(Y)$ for each i, by making a choice of null homotopy for each cell. But then this may not produce a map of towers. That is, the best we can say is that we have a homotopy $g_i \circ f_i^X \simeq f_i^Y \circ g_{i+1}$. Similarly, we can build maps $h_i : P_i(Y) \longrightarrow P_i(Z)$ by making many choices, and we can also build maps $(h \circ g)_i : P_i(X) \longrightarrow P_i(Z)$. But in general, the best we can hope for is a homotopy $(h \circ g)_i \simeq h_i \circ g_i$.

On the other hand, using Version II produces much better results. The map $g_i : P_i(X) \longrightarrow P_i(Y)$ defines itself, and we get a map of towers, so that $g_i \circ f_i^X = f_i^Y \circ g_{i+1}$. Similarly, we get maps h_i and $(hg)_i$ without making any choices, and the functoriality equation $(hg)_i = h_i \circ g_i$ holds.

Question. To what extent can we recover X from its Postnikov tower? Is X equivalent to $\varprojlim P_i(X)$?

Proposition 1.1. *For any sequence of fibrations* $\ldots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0$*, the canonical maps* $\lambda_i : \pi_i(\varprojlim X_n) \longrightarrow \varprojlim \pi_i(X_n)$ *are surjective. Moreover,* λ_i *is injective if* $\pi_{i+1}(X_n) \longrightarrow \pi_{i+1}(X_{n-1})$ *is surjective for sufficiently large* n.

Proof. (Surjectivity) Let $(\gamma_n) \in \varprojlim \pi_i(X_n)$. So for all n, we have a homotopy $h_n : q_{n+1} \circ \gamma_{n+1} \simeq \gamma_n$. Since q_{n+1} is a fibration by assumption, we can lift in the diagram

$$\begin{tikzcd} S^i \arrow[r, "\gamma_{n+1}"] \arrow[d] & X_{n+1} \arrow[d, "q_{n+1}"] \\ S^i \wedge I_+ \arrow[r, "h"] \arrow[ur, dashed] & X_n \end{tikzcd}$$

The lift at time 1 then provides a map γ'_{n+1} (in the same homotopy class as γ_n satisfying $q_{n+1} \circ \gamma'_{n+1} = \gamma_n$. We do this for all n, starting with $n = 0$. This provides a map $S^i \longrightarrow \varprojlim X_n$ which maps to (γ_n).

(Injectivity) Suppose now that $\pi_{i+1}(q_k)$ is surjective for $k > n$. Suppose given $S^i \xrightarrow{\gamma} \varprojlim X_n$ such that $\lambda_i(\gamma) = 0$. Thus $S^i \longrightarrow \varprojlim X_n \longrightarrow X_n$ is null for each n. Let $\delta_n : D^{i+1} \longrightarrow X_n$ be a null-homotopy for the composite. We then get a lift in the diagram

$$\begin{tikzcd} S^i \arrow[r, "\gamma_{n+1}"] \arrow[d] & X_{n+1} \arrow[d, "q_{n+1}"] \\ S^i \wedge I_+ \arrow[r, "\delta_n"'] \arrow[ur, dashed, "h"] & X_n \end{tikzcd}$$

Thus h_1 has image in $F(q_{n+1})$. Since $\pi_{i+1}(q_{n+1})$ is surjective, the map $\pi_i(F(q_{n+1})) \longrightarrow \pi_i(X_{n+1})$ is injective. But $q_{n+1} \circ h_1$ is constant, so h_1 is null in $\pi_1(F(q_{n+1}))$. Writing

$$D^{i+1} \cong (S^i \times I) \cup_{S^i \times \{1\}} D^{i+1} \times \{1\},$$

we may glue the homotopy h to a null-homotopy for h_1 to define a map $\delta_{n+1} : D^{i+1} \longrightarrow X_{n+1}$ lying over δ_n. In the same way, we obtain a null-homotopy $\delta_{n+2} : D^{i+1} \longrightarrow X_{n+2}$ lying over δ_{n+1}, and so on. Taking all of the maps δ_k together provides a map $D^{i+1} \longrightarrow \varprojlim X_n$ exhibiting a null-homotopy for γ. ■

2. Wed., Apr. 6

By the result at the end of last time, if we replace all of the maps in the Postnikov tower by fibrations, then the natural map $f : X \longrightarrow \varprojlim P_i(X)$ is a weak equivalence. Incidentally, the limit of a tower obtained by first replacing all of the maps by fibrations is called the **homotopy limit** of the tower. So we have $X \simeq \operatorname{holim} P_i(X)$.

As we said earlier, Postnikov towers should be viewed as dual to CW complexes. The sort of duality we have in mind is sometimes called Eckmann-Hilton duality. CW complexes are built out of cells (spheres) using cofiber sequences, and Postnikov towers are built out of cocells (Eilenberg-Mac Lane spaces) using "cocells".

In what sense are spheres dual to Eilenberg-Mac Lane spaces? A space X is equivalent to $K(A, n)$ if and only if

$$[S^i, X]_* \cong \begin{cases} A & i = n \\ 0 & i \neq n. \end{cases}$$

Similarly, a (simply connected) space X is equivalent to S^n if and only if

$$[X, K(A, n)]_* \cong \begin{cases} A & i = n \\ 0 & i \neq n. \end{cases}$$

Example 2.1. (1) $X = K(A, n)$. Then of course $P_i(X) = *$ if $i < n$, and $P_i(X) = X$ for $i \geq n$.

(2) $X = K(A, n) \times K(B, m)$ with $m < n$. Then similarly $P_i(X) = *$ for $i < m$. We have $P_i(X) = K(B, M)$ for $m \leq i < n$ and $P_i(X) = X$ for $i \geq n$.

Granted, these are not terribly exciting examples. Unfortunately, it is difficult to find examples of spaces with Postnikov towers that are simultaneously more interesting and completely understood, for the following reason. Any simply connected finite complex X that is not contractible has infinitely many nontrivial homotopy groups. So any finite Postnikov stage must be an infinite complex.

We have discussed the fact that X can be recovered from its Postnikov tower. What data is needed to build the tower? Certainly, knowledge of the homotopy groups is not enough. For the first Postnikov section $P_1(X)$, we only need to know the fundamental group (let's assume X is connected). But the space $P_2(X)$ sits in a fiber sequence $K(\pi_2(X), 2) \longrightarrow P_2(X) \longrightarrow P_1(X)$, and the groups $\pi_1(X)$ and $\pi_2(X)$ do not alone determine the space $P_2(X)$.

Suppose that the fiber sequence extends to the right, so that we have a fiber sequence

$$P_2(X) \longrightarrow P_1(X) \xrightarrow{k_1} K(\pi_2(X), 3)$$

for some map k_1. Then the space $P_2(X)$ would be described by the group $\pi_1(X)$ and by a cohomology class $k_1 \in \mathrm{H}^3(P_1(X); \pi_2(X))$.

This leads to the question of when fiber sequences extend to the right in this way. We say that a fiber sequence $F \longrightarrow E \longrightarrow B$ is a **principal fibration** if the fibration is pulled back from a path loop fibration. That is, we assume a diagram

$$\begin{tikzcd} F \arrow[r, "\sim"] \arrow[d] & \Omega Z \arrow[d] \\ E \arrow[r] \arrow[d] & PZ \simeq * \arrow[d] \\ B \arrow[r] & Z \end{tikzcd}$$

in which the bottom square is a (homotopy) pullback diagram. We will give an equivalent description of principal fibrations shortly, but let us first recall some background. Recall that any path $\gamma : b_0 \longrightarrow b_1$ in B lifts to a homotopy equivalence $F(b_0) \xrightarrow{\tilde{\gamma}} F(b_1)$. In particular, a loop in B gives a self homotopy equivalence of F. If F is simply connected, this gives a well-defined action of $\pi_1(B)$ on $\pi_n(F)$ for all n.

Proposition 2.2. *Let E and B be connected. A fiber sequence $K(A, n) \longrightarrow E \xrightarrow{f} B$ is a principal fiber sequence if and only if $\pi_1(B)$ acts trivially on $\pi_n(K(A, n)) \cong A$.*

Proof. ($\Rightarrow$) Suppose the fibration is principal. Then we can identify the action of $\pi_1(B)$ on $\pi_n(K(A, n))$ in the given fibration with the action of $\pi_1(B)$ on $\pi_{n+1}K(A, n+1)$ in the fibration $E \longrightarrow B \longrightarrow K(A, n+1)$. But this factors through the action of $\pi_1(K(A, n+1)) \cong 0$, so the action must be trivial.

($\Leftarrow$). The map f is an n-equivalence, so the map

$$F(f) \longrightarrow \Omega C(f)$$

is an $(n-1)$-equivalence. But $F(f)$ is $(n-1)$-connected, so $\Omega C(f)$ must also be $(n-1)$-connected, so that $C(f)$ must be n-connected. The Hurewicz theorem now tells us that

$$\pi_{n+1}(C(f)) \cong \tilde{\mathrm{H}}_{n+1}(C(f)) \cong \mathrm{H}_{n+1}(B, E).$$

We need a stronger result, namely a relative Hurewicz theorem. To state it , note that given a pair (X, A), there is an action of $\pi_1(A)$ on $\pi_n(X, A)$ (see Hatcher, p. 345).

Theorem 2.3 (Relative Hurewicz)**.** *Suppose (X, A) is an $(n-1)$-connected pair $(n > 2)$. Then there is a short exact sequence*

$$0 \longrightarrow \{\textit{subgroup generated by } [\gamma \cdot f] - [f]\} \longrightarrow \pi_n(X, A) \longrightarrow \mathrm{H}_n(X, A) \longrightarrow 0$$

■

In particular, if $\pi_1(A)$ acts trivially on $\pi_n(X, A)$, then the relative Hurewicz map $\pi_n(X, A) \longrightarrow \mathrm{H}_n(X, A)$ is an isomorphism. We now have

$$\pi_{n+1}(C(f)) \cong \mathrm{H}_{n+1}(B, E) \cong \pi_{n+1}(B, E) \cong \pi_n(F) = A.$$

Since $C(f)$ is n-connected, we may attach cells of dimension $n+3$ and higher to build a $K(A, n+1)$. Denote by k the composite

$$k : B \longrightarrow C(f) \longrightarrow K(A, n+1).$$

Replacing k by a fibration k', we get the following diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & B \arrow[d, "\simeq"] \arrow[dr, "k"] & \\ F(k') \arrow[r] & B' \arrow[r, "k'"] & K(A, n+1). \end{tikzcd}$$

It remains to check that $E \longrightarrow F(k')$ is an equivalence. Consider the long exact sequences in homotopy

$$\begin{tikzcd}
0 \arrow[r] & \pi_{n+1}(E) \arrow[r] \arrow[d] & \pi_{n+1}(B) \arrow[r] \arrow[d, "\cong"] & A \arrow[r] \arrow[d, "\cong"] & \pi_n(E) \arrow[r] \arrow[d] & \pi_n(B) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \pi_{n+1}(F(k')) \arrow[r] & \pi_{n+1}(B') \arrow[r] & \pi_{n+1}(K(A, n+1)) \arrow[r] & \pi_n(F(k')) \arrow[r] & \pi_n(B') \arrow[r] & 0
\end{tikzcd}$$

We are now done by the 5-lemma.

3. Fri, Apr. 8

The upshot of the discussion last time is

Proposition 3.1. *X has a Postnikov tower <u>of principal fibrations</u> if and only X is simple.*

If this is the case, the maps $P_i(X) \xrightarrow{k_i} K(\pi_{i+1}, i+2)$ correspond to cohomology classes $k_i \in \mathrm{H}^{i+2}(X; \pi_{i+1}(X))$, which are known as the "k-invariants of X."

Obstruction Theory Revisited:

Let X be a CW complex and Y be simple. The construction of maps $X \longrightarrow Y$ is governed by obstruction theory. Since Y is simple, it has a Postnikov tower of principal fibrations. We may therefore build a map $f : X \longrightarrow Y$ by building a compatible family of maps $f^i : X \longrightarrow P_i(Y)$. As before, we build maps $X \longrightarrow P_i(Y)$ inductively over the skeleta of X. Let us assume given a commutative diagram as follows:

$$\begin{tikzcd}
X_n \arrow[r, "f_n"] \arrow[d, "i_n"'] & Y \arrow[r] \arrow[dr] & P_i(Y) \arrow[d] & \\
X_{n+1} \arrow[rr, "f_{n+1}^{i-1}"'] & & P_{i-1} \arrow[r, "k_{i-1}"] & K(\pi_i(Y), i+1)
\end{tikzcd}$$

When have an exact sequence

$$[X_{n+1}, P_i Y] \longrightarrow [X_{n+1}, P_{i-1} Y] \longrightarrow \mathrm{H}^{i+1}(X_{n+1}, \pi_i Y).$$
$$f_{n+1}^{i-1} \quad \mapsto \quad o(f_{n+1}^{i-1})$$

The class $o(f_{n+1}^{i-1})$ is an obstruction class, and if it vanishes we can lift f_{n+1}^{i-1} to a map

$$f_{n+1}^{i} : X_{n+1} \longrightarrow P_i Y.$$

Of course, we would like this lift to be compatible with f_n, so we want a lift rel X_n.

The commutativity of the diagram tells us that $o(f_{n+1}^{i-1}) \mid_{X_n} = 0$, so the obstruction class lifts to a relative obstruction class

$$o(f_{n+1}^{i-1}, X_n) \in \mathrm{H}^{i+1}(X_{n+1}, X_n; \pi_i Y)$$

which controls relative lifts. Note that this group vanishes unless $i = n$. Thus there is a single layer in the Postnikov tower at which lifting is nontrivial.

Next, we will discuss an important family of spaces whose k-invariants are always trivial. This is equivalent to saying that the space is weakly equivalent to a product of Eilenberg-Mac Lane spaces. Such a space is sometimes referred to as a Generalized Eilenberg-Mac Lane space, or GEM for short.

Theorem 3.2 (J. Moore)**.** *A topological abelian monoid X is a GEM:*

$$X \simeq \prod_{n \geq 0} K(\pi_n(X), n).$$

Proof. Recall that for each abelian group A and $n \geq 0$, there is a Moore space $M(A,n)$ with

$$\tilde{\mathrm{H}}_i(M(A,n);\mathbb{Z}) \cong \begin{cases} A & i = n \\ 0 & i \neq n. \end{cases}$$

This space is built as a cofiber

$$\bigvee S^n \longrightarrow \bigvee S^n \longrightarrow M(A,n).$$

For each n, it is easy to build a map

$$M(\pi_n(X),n) \xrightarrow{\theta_n} X$$

giving a π_n-isomorphism (by Hurewicz, $\pi_n(M(A,n)) \cong \mathrm{H}_n(M(A,n);\mathbb{Z}) \cong A$).

Now for any space Y, the space $B(Y,\mathbb{N})$ is a topological abelian monoid[1] — in fact, it is the free topological abelian monoid on Y. Any map of spaces $Y \longrightarrow X$ extends to a map of topological monoids $B(Y,\mathbb{N}) \longrightarrow X$. Another way to say this is that the functor $B(-,\mathbb{N})$ sits in an adjoint pair

$$B(-,\mathbb{N}) : \mathbf{Top} \rightleftarrows \mathbf{TopAbMon} : U,$$

where U is the forgetful functor. We need the basepointed version, which sits in an adjoint pair

$$\tilde{B}(-,\mathbb{N}) : \mathbf{Top}_* \rightleftarrows \mathbf{TopAbMon} : U,$$

Recall, for example, that $\tilde{B}(S^n,\mathbb{Z})$ is a model for $K(\mathbb{Z},n)$. It turns out[2] that the natural map $\tilde{B}(Y,\mathbb{N}) \longrightarrow \tilde{B}(Y,\mathbb{Z})$ is a "topological group completion". This means, in particular, that the map is a weak equivalence if Y is connected. Moreover, the earlier statement that $\tilde{B}(S^n,\mathbb{Z}) \simeq K(\mathbb{Z},n)$ can be generalized, using the fact that $\tilde{B}(-,\mathbb{Z})$ converts cofiber sequences to fiber sequences, to the statement that $\tilde{B}(M(A,n),\mathbb{Z}) \simeq K(A,n)$. Putting these results together, we learn that $\tilde{B}(M(A,n),\mathbb{N}) \simeq K(A,n)$.

We may assemble the maps θ_n to get a map of spaces

$$\bigvee_{n\geq 0} M(\pi_n(X),n) \longrightarrow X.$$

This corresponds to a map of topological abelian monoids

$$\tilde{B}\left(\bigvee_{n\geq 0} M(\pi_n(X),n),\mathbb{N}\right) \longrightarrow X.$$

The domain can be identified as

$$\tilde{B}\left(\bigvee_{n\geq 0} M(\pi_n(X),n),\mathbb{N}\right) \cong \bigoplus_n \tilde{B}(M(\pi_n(X),n),\mathbb{N}) \simeq \bigoplus K(\pi_n(X),n).$$

Note that the natural map

$$\bigoplus_{n\geq 0} K(\pi_n(X),n) \longrightarrow \prod_{n\geq 0} K(\pi_n(X),n)$$

is an equivalence since each factor has homotopy in a single degree.

As we have said, each factor induces an isomorphism

$$\pi_n(K(\pi_n(X),n)) \xrightarrow{\cong} \pi_n(X),$$

[1]By $\mathbb{N}$, we really mean the monoid $\mathbb{Z}_{\geq 0}$

[2]This is the Dold-Thom Theorem.

so the map

$$\prod_{n\geq 0} K(\pi_n(X), n) \xrightarrow{\sim} X$$

is an equivalence. ■

This result says that topological abelian monoids (and therefore also topological abelian groups) are completely determined by their homotopy groups.

The space $B(Y, \mathbb{N})$ from above is also known as the "infinite symmetric product" and denoted $\mathrm{SP}(Y)$ or $\mathrm{Symm}(Y)$ (see Hatcher, Appendix 4K). This is the union $\mathrm{Symm}(Y) = \bigcup_n \mathrm{Symm}_n(Y)$, where

$$\mathrm{Symm}_n(Y) = Y^n/\Sigma_n \cong \{ \text{ unordered } n\text{-tuples of points in } Y\}.$$

For example, we think of the class of $(y_1, y_3, y_1, y_1, y_3, y_2)$ as representing $3y_1 + y_2 + 2y_3$.

Example 3.3. Let $Y = \mathbb{CP}^1 \cong S^2$. We claim that $\mathrm{Symm}_n(S^2) \cong \mathbb{CP}^n$. To see this, first note that we can identify $\mathbb{CP}^n$ with the set of nonzero $\mathbb{C}$ polynomials of degree at most n, up to scalar multiple. The correspondence identifies the point $[y_0 : \cdots : y_n]$ of $\mathbb{CP}^n$ with $f(t) = y_n t^n + y_{n-1}t^{n-1} + \cdots + y_0$. Now define

$$(\mathbb{CP}^1)^n \longrightarrow \mathbb{CP}^n$$

by

$$(z_1, \ldots, z_n) \mapsto f(t) = \prod_i (t - z_i).$$

If any of the coordinates is $z_i = \infty$, the corresponding factor $t - z_i$ is omitted from $f(t)$. This function is clearly symmetric and so induces a map

$$\mathrm{Symm}_n(\mathbb{CP}^1) \longrightarrow \mathbb{CP}^n.$$

The fact that polynomials are determined by their roots means that this map is a bijection. Both spaces are compact Hausdorff, so it is a homeomorphism as well. It then follows that $\mathrm{Symm}(S^2) \cong \mathbb{CP}^\infty \simeq K(\mathbb{Z}, 2)$.