CLASS NOTES
MATH 527 (SPRING 2011)
WEEK 7

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1. Mon, Feb. 21

NO CLASS

2. Wed, Mar. 2

More examples

(5) The Hopf fibration $\eta : S^3 \longrightarrow S^2$ is an S^1-bundle (with fiber S^1, so this is a *principal* S^1-bundle)

(6) The higher Hopf maps: $\nu : S^7 \longrightarrow S^4$ and $\sigma : S^{15} \longrightarrow S^8$. These are made by the same construction as for η, using the quaternions or octonions, respectively, instead of $\mathbb{C}$. The map ν is a principal S^3-bundle. Recall that S^3 is the group of unit quaternions and can be identified with the Lie group $SU(2)$.

The unit octonions S^7 do not form a group since the octonions are not associative. Nevertheless, σ is a fiber bundle with fiber S^7.

There is one more fibration we have not mentioned: if we use $\mathbb{R}$ instead of $\mathbb{C}$, we get the quotient map $S^1 \longrightarrow \mathbb{RP}^1$. But $\mathbb{RP}^1 \cong S^1$, and this map is a double cover. So the Hopf map $S^1 \longrightarrow S^1$ is just the degree 2 map. It is a consequence of the Hopf invariant one problem, solved by Frank Adams, that there are no other fiber sequences in which the base, total space, and fiber are all spheres.

If $f : E \longrightarrow B$ is based, we can identify the point-set fiber of the fibration $q : P(f) \longrightarrow B$ with $F(f)$. More generally, we have

Proposition 2.1. *If $f : E \longrightarrow B$ is a fibration, then $f^{-1}(*) \simeq F(f)$.*

Proof. This is the dual of the argument for homotopy equivalence of the quotient of a cofibration with the cofiber.

Let $E \xrightarrow{p} B$ be a fibration and consider the following test diagram:

$$\begin{tikzcd}
F(f) \arrow[r, "p"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
F(f) \times I \arrow[r, "ev"'] \arrow[ur, dashed, "\tilde{h}"] & B.
\end{tikzcd}$$

By the definition of $F(f)$, the map $\tilde{h}_1 : F(f) \longrightarrow E$ factors through $f^{-1}(*)$. We also have the natural map $\alpha : f^{-1}(*) \longrightarrow F(f)$ defined by $\alpha(e) = (e, c_*)$ (we are writing $*$ for the basepoint in B). Then $\tilde{h}_1 \circ \alpha(e)$ is the endpoint of a path in $f^{-1}(b_0)$ starting at e. A homotopy back to e is given by traveling along this path. The homotopy $H : \mathrm{id} \simeq \alpha \circ \tilde{h}_1$ is given by

$$H((e,\gamma),t) = (\tilde{h}((e,\gamma),t), \gamma|_{[t,1]}).$$

■

Actually, the preceding result follows from the following statement, which is dual to the statement that if $A \xrightarrow{f} X$ is based and a homotopy equivalence and A and X are well-pointed, then f is a based homotopy equivalence:

Proposition 2.2. *If $E \longrightarrow B$ is a fibration, then the map $E \xrightarrow{j} P(f)$ is a homotopy equivalence over B .*

The homotopy type of the homotopy fiber does not depend on the choice of basepoint of B (within a given path-component of B):

Proposition 2.3. *Let γ be a path in B, and write $b_0 = \gamma(0)$, $b_1 = \gamma(1)$. Then if $p : E \longrightarrow B$ is a fibration, the fibers $F_0 = p^{-1}(b_0)$ and $F_1 = p^{-1}(b_1)$ are homotopy equivalent.*

Proof. We define maps between F_0 and F_1 as the time 1 lifts in the diagrams

$$\begin{tikzcd} F_0 \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ F_0 \times I \arrow[r, "\gamma"] \arrow[ur, dashed, "\alpha"] & B \end{tikzcd} \qquad \begin{tikzcd} F_1 \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ F_1 \times I \arrow[r, "\gamma^{-1}"] \arrow[ur, dashed, "\beta"] & B. \end{tikzcd}$$

It remains to show $\alpha_1 \circ \beta_1 \simeq \mathrm{id}$ and $\beta_1 \circ \alpha_1 \simeq \mathrm{id}$.

We have homotopies $F_0 \times I \longrightarrow E$ given by

$$\beta \circ (\alpha_1 \times \mathrm{id}) : \alpha_1 \simeq \beta_1 \circ \alpha_1$$

and

$$\alpha : \mathrm{id} \simeq \alpha_1.$$

We the concatenation of these homotopies is thus a homotopy $H : F_0 \times \longrightarrow E$ from id to $\beta_1 \circ \alpha_1$, but it does not lift to a homotopy $F_0 \times I \longrightarrow F_0$. We can fix this as follows: let $G : F_0 \times I^2 \longrightarrow B$ represent a homotopy $c_{b_0} \simeq \gamma^{-1} * \gamma$ through loops at b_0. We then get a lift in the diagram

$$\begin{tikzcd} F_0 \times I \arrow[r, "H"] \arrow[d] & E \arrow[d, "p"] \\ F_0 \times I^2 \arrow[r, "G"'] \arrow[ur, dashed, "K"] & B. \end{tikzcd}$$

Using a homeomorphism $J^2 \cong I$, the restriction of K to $F_0 \times J^2$ gives a homotopy $F_0 \times I \longrightarrow E$ living over the constant map to $b_0 \in B$. In other words, we get a homotopy $\mathrm{id} \simeq \beta_1 \circ \alpha_1$.

A similar argument gives a homotopy $\alpha_1 \circ \beta_1$ on F_1. ■

Our above discussion gives a long exact sequence in homotopy arising from a fibration, but this can be improved to a statement about Serre fibrations:

Proposition 2.4. *Let $p : E \longrightarrow B$ be a Serre fibration. Then $\pi_n(p^{-1}(b_0)) \cong \pi_{n+1}(B, E)$.*

We will prove this next time.

Homotopy pullbacks

Dual to the notion of homotopy pushout is that of a homotopy pullback. One can define the homotopy pullback of a pair of maps $A \xrightarrow{f} B \xleftarrow{g} C$ to be "double path space", by which we mean the pullback in the diagram $P(f) \to B \leftarrow P(g)$. It is common to write $A \times_B^h C$ for the homotopy pullback, and the maps $A \longrightarrow P(f)$ and $C \longrightarrow P(g)$ provide a map

$$A \times_B C \longrightarrow A \times_B^h C.$$

There are two versions of homotopy pullbacks: one is homotopy invariant, while the other is weak homotopy invariant.

Proposition 2.5. *Suppose given a pair of maps $A \xrightarrow{f} B \xleftarrow{g} C$.*

(1) *If either f or g is a fibration, then the map $A \times_B C \longrightarrow A \times_B^h C$ is a homotopy equivalence.*
(2) *If either f or g is a Serre fibration, then the map $A \times_B C \longrightarrow A \times_B^h$ is a weak homotopy equivalence.*

3. Fri, Mar. 4

Leftover from last class:

Proposition 3.1. *Let $p : E \longrightarrow B$ be a Serre fibration. Then $\pi_n(p^{-1}(b_0)) \cong \pi_{n+1}(B, E)$.*

Proof. As we already know that $\pi_n(F(p)) \cong \pi_{n+1}(B, E)$, it suffices to show that the relative homotopy groups $\pi_n(F(p), p^{-1}(b_0))$ vanish and that $\pi_0(p^{-1}(b_0)) \longrightarrow \pi_0(F(p))$ is a bijection. We deal with the π_0 statement first. Let $(e, \gamma) \in F(p)$. Let $\tilde{\gamma}$ be a lift of γ with $\tilde{\gamma}(0) = e$. We claim that (e, γ) and $(\tilde{\gamma}(1), c_{b_0})$ lie in the same path-component of $F(p)$. Indeed, a path between them is given by

$$t \mapsto (\tilde{\gamma}(t), \gamma|_{[t,1]}).$$

Suppose given a map of triples $(I^n, \partial I^n, J^n) \xrightarrow{(f,g)} (F(p), p^{-1}(b_0), e_0)$. This corresponds to a map of triples

$$(I^{n+1}, I^n \times \{0\}, J^n \times \{0\}) \xrightarrow{(F,G,*)} (B, E, e_0)$$

such that the restriction of F to $J^{n+1} \subseteq I^n \times I$ is constant at the basepoint b_0.

It is then easy to find a homotopy H from $F : I^n \times I \longrightarrow B$ to the constant map $I^n \times I \longrightarrow B$. In fact, given that $F|_{J^{n+1}}$ is constant at b_0, we may take the homotopy H to be defined by F on the face $I^n \times \{0\} \times I$. We wish to lift this to a map to E with initial data G. This lift will also be required to be constant along $J^n \times \{0\} \times I$. The desired lift arises from the following lifting diagram

$$\begin{tikzcd} \big(I^n \times \{0\} \times \{0\}\big) \cup \big(J^n \times \{0\} \times I\big) \arrow[r] \arrow[d] & E \arrow[d] \\ I^n \times \{0\} \times I \arrow[r, "F"] \arrow[ur, dashed] & B. \end{tikzcd}$$

A lift exists because the top left space is homeomorphic to $I^n \times \{0\} \times \{0\}$. ■

Back to homotopy pullbacks: In any square

$$\begin{tikzcd} D \arrow[r] \arrow[d] & C \arrow[d] \\ A \arrow[r] & B, \end{tikzcd}$$

there are always maps

$$D \longrightarrow A \times_B C \longrightarrow A \times_B^h C,$$

and we say that the given square is a homotopy pullback square (or is homotopy cartesian) if the above composition is a homotopy equivalence. We say the square is a weak homotopy pullback square if the composition is a weak homotopy equivalence.

Remark 3.2. The term "homotopy pullback square" in the literature is used to mean either of the above two situations, but it more often refers to what we are calling a weak homotopy pullback.

p-cartesian squares and the Homotopy Excision Theorem

Recall that $(X; A, B)$ is an excisive triad if $A, B \subseteq X$ and X is the union of the interiors of A and B.

Theorem 3.3 (Blakers-Massey Homotopy Excision Theorem)**.** *Let $(X; A, B)$ be an excisive triad with $C = A \cap B$ nonempty. Then, if the inclusions $C \hookrightarrow A$ and $C \hookrightarrow B$ are p-connected and q-connected, respectively, then the square*

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

is $(p + q - 1)$-cartesian, meaning that the map $C \longrightarrow B \times^h_X A$ is a $(p + q - 1)$-equivalence.

The following result gives another interpretation of k-cartesian squares.

Proposition 3.4. *Consider a commutative square*

$$\begin{tikzcd} D \arrow[r, "g'"] \arrow[d, "f'"'] & C \arrow[d, "f"] \\ B \arrow[r, "g"'] & A. \end{tikzcd}$$

Then

1. *The square is a weak homotopy pullback square if and only if for every $b \in B$, the induced map $g'' : F_b(f') \longrightarrow F_{g(b)}(f)$ is a weak homotopy equivalence*
2. *The square is k-cartesian if and only if for every $b \in B$, the induced map $g'' : F_b(f') \longrightarrow F_{g(b)}(f)$ is a k-equivalence.*

Proof. Without loss of generality, we may assume that f is a fibration, so that $F_{g(b)}(f) = f^{-1}(g(b))$. Then in the diagram

$$\begin{tikzcd} F(f') \arrow[r, "\Phi"] \arrow[d] & p^{-1}(b) \arrow[r, "\cong"] \arrow[d] & f^{-1}(g(b)) \arrow[d] \\ D \arrow[r, "\Lambda"] \arrow[d, "f'"'] & B \times^h_A C \arrow[r] \arrow[d, "p"'] & C \arrow[d, "f"] \\ B \arrow[r, equal] & B \arrow[r, "g"'] & A, \end{tikzcd}$$

the bottom right square is strict pullback, and the map $p^{-1}(b) \longrightarrow f^{-1}(g(b))$ is a homeomorphism. The 5-lemma, combined with the long exact sequence in homotopy for the fibrations

$$F(f') \longrightarrow D \longrightarrow B$$

and

$$p^{-1}(b) \longrightarrow B \times^h_A C \longrightarrow B$$

imply that Λ is a weak equivalence $\iff$ Φ is a weak equivalence $\iff$ g' is a weak equivalence. The same argument shows that Λ is a k-equivalence if and only if g' is a k-equivalence. ∎

Before giving the proof of the Blakers-Massey Theorem, we will deduce a number of important consequences.

Theorem 3.5. *(Freudenthal Suspension Theorem) Let $E : \pi_j(X) \longrightarrow \pi_{j+1}(\Sigma X)$ be the suspension map*

$$\pi_j(X) = [S^j, X]_* \longrightarrow [S^1 \wedge S^j, S^1 \wedge X]_* \cong [S^{j+1}, \Sigma X]_* = \pi_{j+1}(\Sigma X).$$

If X is well-pointed and n-connected, this map is an isomorphism if $j \leq 2n$ and is a surjection for $j = 2n + 1$.

Proof. Consider the homotopy pushout square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X \end{tikzcd}$$

(need X-well-pointed to deduce the inclusion $X \hookrightarrow C_*(X)$ is a cofibration). Since X is n-connected and $C_*(X)$ is contractible, the inclusions $X \hookrightarrow CX$ are $(n+1)$-equivalences. It follws that the square is $(2n+1)$-cartesian, so that $X \longrightarrow \Omega\Sigma(X)$ is a $(2n+1)$-equivalence. Thus the induced map

$$\pi_j(X) \longrightarrow \pi_j(\Omega\Sigma X) \cong \pi_{j+1}(\Sigma X)$$

is an isomorphism for $j \leq 2n$ and a surjection for $j = 2n+1$. ∎