MATH 651
HOMEWORK II
PARTIAL SOLUTIONS
SPRING 2013

Problem 6. A **free loop** in a space X is a map $S^1 \longrightarrow X$ with no condition on the basepoint. Since $\pi_1(X, x_0) \cong [S^1, (X, x_0)]_*$, there is a natural map

$$\Lambda : \pi_1(X, x_0) \longrightarrow [S^1, X].$$

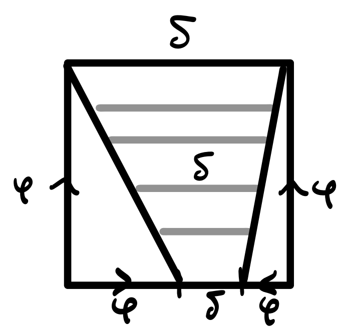
(1) Show that Λ is surjective if and only if X is path-connected.
(2) Show that $\Lambda([\gamma]) = \Lambda([\delta])$ if and only if the loops γ and δ are conjugate in $\pi_1(X, x_0)$.

Solution. Recall that $S^1 \cong [0,1]/\{0,1\}$. It will be convenient to go back and forth between maps $S^1 \longrightarrow X$ and maps $[0,1] \longrightarrow X$ which agree at the endpoints. We can use this for homotopies as well: a homotopy $h : S^1 \times I \longrightarrow X$ is equivalent to a homotopy $h' : I \times I \longrightarrow X$ such that $h'(0,t) = h'(1,t)$ for every t.

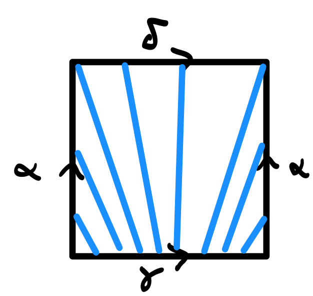
(1) ($\Rightarrow$) Suppose that Λ is surjective. Let x_1 be a point in X. Then there is a loop γ at x_0 such that $\Lambda([\gamma]) = [c_{x_1}]$. In other words, there is a homotopy $h : \gamma \simeq c_{x_1}$. Then $h(0,t)$ defines a path in X from x_0 to x_1.

($\Leftarrow$) Assume that X is path connected, and let $f : S^1 \longrightarrow X$ be any map. We write δ for the resulting path $[0,1] \longrightarrow S^1 \longrightarrow X$ and x_1 for the basepoint of δ. Let φ be any path from x_0 to x_1. Then the path-composite $\varphi * (\delta * \varphi^{-1})$ is a loop at x_0. Write γ for this loop. It remains to show that $\Lambda([\gamma]) = [\delta]$. In other words, we must show that $\gamma \simeq \delta$ as maps $S^1 \longrightarrow X$. A homotopy is given by

$$h(s,t) = \begin{cases} \varphi(t + 2s) & 0 \le 2s \le 1 - t \\ \delta\left(\frac{4s-2+2t}{1+3t}\right) & 2 - 2t \le 4s \le 3 + t \\ \varphi(4(1-s) + t) & 3 + t \le 4s \le 4. \end{cases}$$



(2) ($\Rightarrow$) Suppose that $\Lambda([\gamma]) = \Lambda([\delta])$. Then there is a homotopy $h : \gamma \simeq \delta$ as maps $S^1 \longrightarrow X$. Note that $h(0,t)$ defines a loop φ at x_0. Since h is a homotopy through maps out of the circle, $h(1,t)$ is the same loop φ. We want to show that $[\gamma] = [\varphi * \delta * \varphi^{-1}]$. A path-homotopy is illustrated in the figure to the right.



Date: January 30, 2013.

Said differently, let $i_0 : I \longrightarrow I \times I$ be the inclusion as the bottom edge and let $i_1 : I \longrightarrow I \times I$ be a path that first travels up the vertical left edge, then across the top edge, then down the right edge. By the definition of h, the path γ is $h \circ i_0$ and the path $\varphi * \delta * \varphi^{-1}$ is $h \circ i_1$. But the two maps i_0 and i_1 are certainly path-homotopic, as $I \times I$ is contractible (even convex). It follows that $\gamma \simeq_p \varphi * \delta * \varphi^{-1}$.

($\Leftarrow$) Assume $h : \gamma \simeq_p \varphi * \delta * \varphi^{-1}$ is a path-homotopy for some loop φ. Since h is a path-homotopy, it must be constant on the vertical edges of the square $I \times I$. Collapsing each of these edges produces a disc D^2, and we get an induced map $\overline{h} : D^2 \longrightarrow X$. The desired free homotopy $H : \gamma \simeq \delta$ is given by the following composition, where the first map is any homeomorphism identifying the boundary as indicated.