# Math 651 - Topology II
# Homework X
# Spring 2014

1. Recall that the **rank** of an abelian group is the maximal number of linearly independent elements in the group. If $A \cong \mathbb{Z}^r \times B$, where $B$ is a finite group, then $\operatorname{rank} A = r$.

   A chain complex is said to be **exact** if all homology groups are zero. An exact chain complex of the form

   $$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

   is called a **short exact sequence**.

   $(a)$ What does exactness at $A$ tell you about the map $f$? What does exactness at $C$ tell you about $g$?

   $(b)$ Show that if $A$, $B$, and $C$ are all finitely generated, then

   $$\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C).$$

2. If $C_*$ and $D_*$ are chain complexes, then a map of complexes $f_* : C_* \longrightarrow D_*$ is a sequence of homomorphisms $g_n : C_n \longrightarrow D_n$ commuting with the differentials:

   $$\begin{tikzcd}
   \cdots \arrow[r] & C_3 \arrow[r, "d_3"] \arrow[d, "f_3"] & C_2 \arrow[r, "d_2"] \arrow[d, "f_2"] & C_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & C_0 \arrow[d, "f_0"] \\
   \cdots \arrow[r] & D_3 \arrow[r] & D_2 \arrow[r, "d_2"] & D_1 \arrow[r, "d_1"] & D_0
   \end{tikzcd}$$

   $(a)$ Show that a map of complexes gives rise to homomorphisms

   $$H_*(f_*) : H_*(C_*) \longrightarrow H_*(D_*).$$

   $(b)$ Given a cellular map $f : X \longrightarrow Y$ of CW complexes, use this to define a map of chain complexes $f_* : C_*(X) \longrightarrow C_*(Y)$.

3. Show that $S^1 \vee S^1 \vee S^2$ has the same homology as $T^2$.