29. Mon, Mar. 31

Last time we were discussing the torus, and we arrived at $\pi_1(T^2) \cong F(a,b)/\langle aba^{-1}b^{-1}\rangle$. Here is a proof that this is isomorphic to $\mathbb{Z}^2$.

Proposition 29.1. *The natural map $\varphi : F(a,b) \longrightarrow \mathbb{Z}^2$ defined by $\varphi(a) = (1,0)$ and $\varphi(b) = (0,1)$ induces an isomorphism*

$$F(a,b)/\langle aba^{-1}b^{-1}\rangle \cong \mathbb{Z}^2.$$

Proof. Let $K = \ker(\varphi)$ and let $N \trianglelefteq F(a,b)$ be the normal subgroup generated by $aba^{-1}b^{-1}$. By the First Isomorphism Theorem, $F(a,b)/K \cong \mathbb{Z}^2$, so it suffices to show that $N = K$. It is clear that $N \leq K$. Since $N \trianglelefteq K$, we wish to show that the quotient group K/N is trivial. Let $g = a^{n_1}b^{k_1}a_{n_2}b^{k_2}a^{n_3} \in K/N$. In K/N, we have $\overline{ab} = \overline{ba}$, so

$$a^{n_1}b^{k_1}a^{n_2}b^{k_2}a^{n_3} = a^{n_1+n_2+n_3}b^{k_1+k_2}.$$

Since $g \in K$, we have $n_1 + n_2 + n_3 = 0$ and $k_1 + k_2 = 0$, so $\overline{g} = e$ in K/N. ■

Our CW structure had a single 0-cell, two 1-cells, and a single 2-cell, so we find that the Euler characteristic is

$$\chi(T^2) = 1 - 2 + 1 = 0.$$

Example 29.2. (Klein bottle) One definition of the Klein bottle K is as the quotient of I^2 in which one opposite pair of edges is identified with a flip, while the other pair is identified without a flip. This leads to the computation

$$\pi_1(K) \cong F(a,b)/\langle aba^{-1}b\rangle.$$

For certain purposes, this is not the most convenient description. Cut the square along a diagonal and repaste the triangles along the previously flip-identified edges. The resulting square leads to the computation

$$\pi_1(K) \cong F(a,c)/\langle a^2c^2\rangle.$$

The equation $c = a^{-1}b$ allows you to go back and forth between these two descriptions.

Like the torus, the resulting cell complex has a single 0-cell, two 1-cells, and a single 2-cell, so

$$\chi(K) = 1 - 2 + 1 = 0.$$

The next example is <u>not</u> obtained by attaching a cell to $S^1 \vee S^1$.

Example 29.3. If we glue the boundary of I^2 according to the relation $abab$, the resulting space can be identified with $\mathbb{RP}^2$. Notice in this case that the four vertices do not all become identified. Rather they are identified in pairs, and we are left with two vertices after making the quotient. This example can be visualized by thinking of identifying the two halves of ∂D^2 via a twist.

These 2-dimensional cell complexes are all examples of **surfaces** (2-dimensional manifolds).

30. Wed, Apr. 2

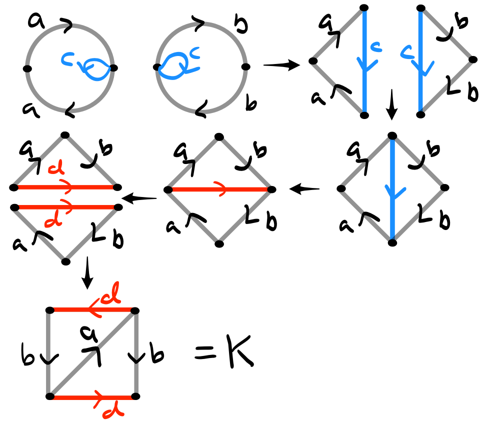
There is an important construction for surfaces called the **connected sum**.

Definition 30.1. Suppose M and N are surfaces. Pick subsets $D_M \subseteq M$ and $D_N \subseteq N$ that are homeomorphic to D^2 and remove their interiors from M and N. Write $M' = M - \mathrm{Int}(D_M)$ and $N' = N - \mathrm{Int}(D_N)$. Then the connected sum of M and N is defined to be

$$M\#N = M' \cup_{S^1} N',$$

where the maps $S^1 \longrightarrow M'$ and $S^1 \longrightarrow N'$ are the inclusions of the boundaries of the removed discs.

For two-dimensional cell-complexes, this can be visualized as in the example to the right, which shows that $\mathbb{RP}^2\#\mathbb{RP}^2 \cong K$, the Klein bottle.



Example 30.2. If M is a surface, then the connect sum $M\#S^2$ is again homeomorphic to M.

Example 30.3. If M is a surface, then the connect sum $M\#T^2$ can be viewed as M with a "handle" glued on.

For example, consider $M = T^2$. Then $T^2\#T^2$ looks liked a "two-holed torus". This is called M_2, the (orientable) surface of genus two. From the cell structure resulting from the picture, we see a wedge of four circles (let's call the generators of the circles a_1, b_1, a_2, b_2) with a two-cell attached along the element $[a_1, b_1][a_2, b_2]$. It follows that the fundamental group of M_2 is

$$F(a_1, b_1, a_2, b_2)/[a_1, b_1][a_2, b_2].$$

We also find that $\chi(M_2) = 1 - 4 + 1 = -2$.

Example 30.4. (Surface of genus g) Similarly, if we take a connect sum of g tori, we get the surface of genus g, M_g. It has fundamental group

$$F(a_1, b_1, \ldots, a_g, b_g)/[a_1, b_1] \ldots [a_g, b_g].$$

We now have $\chi(M_g) = 1 - 2g + 1 = 2 - 2g$.

We are headed towards a "classification theorem" for compact surfaces, so let us now show that if $g_1 \neq g_2$ then M_{g_1} is not homeomorphic to M_{g_2}. We show this by showing they have different fundamental groups. As we have said already, understanding a group given by a list of generators and relations is not always easy, so we make life easier by considering the **abelianizations** of the fundamental groups.

We introduced this already last week, but let's review. The abelianization G_{ab} of G is the group defined by

$$G_{ab} = G/[G, G],$$

where $[G, G]$ is the (normal) subgroup generated by commutators.

Lemma 30.5. *The abelianization $F(a_1, \ldots, a_n)_{ab}$ is the free abelian group $\mathbb{Z}^n$.*

Now the product of commutators $[a_1, b_1] \ldots [a_g, b_g]$ is of course in the commutator subgroup of $F(a_1, b_1, \ldots, a_g, b_g)$, and it follows that $\pi_1(M_g)_{ab} \cong \mathbb{Z}^g$.

Lemma 30.6. *If $H \cong G$ then $H_{ab} \cong G_{ab}$.*

As a result, we see that if $g_1 \neq g_2$ then $\pi_1(M_{g_1}) \neq \pi_1(M_{g_2})$ because their abelianizations are not isomorphic.

Note that we have also distinguished all of these from S^2 (which has trivial fundamental group) and from $\mathbb{RP}^2$ (which has abelian fundamental group $\mathbb{Z}/2\mathbb{Z}$).

31. Fri, Apr. 4

No class (NCUR).